CHAPTER EIGHT

PERCEPTIONS AND REALITIES

In this chapter we discuss the dynamics of perspectives relative to a given participator. Accordingly, we shall imagine this participator to be the first of a fixed number, say k, of participators on a symmetric framework. We then discuss general conditions in which this given participator perceives the dynamical situation truly.

1. Introduction

The study of true perception by a single participator involves a new stochastic process, one which may be arrived at from the augmented dynamics in three stages. First, our participator, call it "A_1," is ignorant of the channeling involutions: we must ***standardize.*** Furthermore, as discussed in earlier chapters, A_1 does not know its absolute perspective: we must ***relativize*** with respect to its perspective. And finally, A_1 only "looks" when it is channelled to: the relevant time parameter of our stochastic process must be the ***proper time*** of A_1. As we shall see, this (random) proper time is a stopping time for the augmented or standard chains we have hitherto studied.

In summary, the primary stochastic process, from which all others studied here derive, is the augmented absolute position chain. To give conditions for which perception matches reality, we are interested in the stochastic process which is obtained from the augmented chain (i) by a ***standardization,*** i.e., in ignorance of the full channeling involutions, (ii) by a ***relativization,*** i.e., in ignorance of absolute perspectives, and (iii) by a ***"trace-operation,"*** i.e., in ignorance of the instants when A_1 is ***not*** channeled to. The first question we wish to address is this: Does this triple succession of information losses yield a Markov chain? We demonstrate in section five that it does, and show there how to compute the transition probabilities of the resulting chain. We then address the implications of this result for true perception in terms of stationary measures for the Markov chains involved.

2. Relativization

Given k participators on a symmetric framework with τ-distribution, we have the associated augmented and standard dynamical chains, which we introduced in chapter seven. We will now consider ***relative dynamics,*** which is intuitively the (standard or augmented) dynamics seen from the viewpoint of one of the participators. Thus, the relative dynamics with respect to, say, the first participator is the standard dynamics in which the positions of the participators are now described in terms of a moving frame which is always centered at the location of the first participator. In **6**–4 we considered a special case of the relative dynamics, namely for two-participator systems in a framework where E is itself an abelian group, and in which there are no self-channelings. ("No self-channelings" is a statement about the τ-distribution.) We were then able to express the relative positions of the participators as a chain in J (which in that chapter equaled E). The transition probability P was found to be obtainable from the action kernels Q, R of the separate participators in terms of a "bracket operation," i.e., $P = [Q, R]$. In this section we will consider a more general case. We consider ***augmented relative dynamics*** which is augmented dynamics represented from one participator's viewpoint. The procedure of passing from either the standard or augmented chains to the corresponding relative dynamics is called ***relativization.*** The motivation for the introduction of symmetric frameworks is that they provide the minimum structure necessary for the relativization of dynamics.

Hypothesis 2.0. As usual we assume that we are in a symmetric framework $\{X, Y, E, S, G, J, \pi\}$ with given configuration symmetric τ-distribution (**7**–2.3) and that we have k participators with symmetric action kernels $Q_1(t), \ldots, Q_k(t)$. Throughout this chapter we assume that E is a principal homogeneous space for J. Define the maps $q\colon E^k \to J^{k-1}$ and $\hat{q}\colon E^k \times \mathcal{I}(k) \to J^{k-1} \times \mathcal{I}(k)$ as follows

$$q(e) = (e_2 e_1^{-1}, \ldots, e_i e_1^{-1}, \ldots, e_k e_1^{-1});$$
$$\hat{q}(e, \chi) = (q(e), \chi); \quad e = (e_1, \ldots, e_k) \in E^k, \ \chi \in \mathcal{I}(k).$$

We denote the space $J^{k-1} \times \mathcal{I}(k)$ by $\hat{J}^{k-1}$. We have the standard dynamical chain in E^k with one-step transition probabilities given by kernels $N_t = \langle Q_1(t), \ldots, Q_k(t) \rangle_\tau$ (Definition **7**–4.1) and the augmented chain in $\hat{E}^k = E^k \times \mathcal{I}(k)$ with kernels $\hat{N}_t = \widehat{\langle Q_1(t), \ldots, Q_k(t) \rangle}_\tau$ (Definition **7**–3.2).

We may express the condition that the τ distribution is configuration symmetric (**7**–2.3) as follows: τ is symmetric iff $\tau(\cdot; \chi)$ is constant on fibres of q, i.e.,

$$q(e) = q(e') \Rightarrow \tau(e; \chi) = \tau(e'; \chi).$$

We shall now see that the relativization procedure results in a markovian dynamics.

Theorem 2.1. $\hat{N}_t$ is $\hat{q}$-respectful. N_t is q-respectful.

Proof. (We drop the subscript t in the sequel.) Let us consider the kernel $\hat{N}$; the proof for N is similar. We are to show that for any e, $e' \in E^k$ such that $q(e) = q(e')$, for all $A \in \mathcal{J}^{k-1}$, and for all χ_0, $\chi_1 \in \mathcal{I}(k)$ the following equality holds:

$$\hat{N}(e, \chi_0; q^{-1}(A) \times \{\chi_1\}) = \hat{N}(e', \chi_0; q^{-1}(A) \times \{\chi_1\}).$$

Recalling Definition **7**–3.2, we have

$$\begin{aligned} &\hat{N}(e, \chi_0; q^{-1}(A) \times \{\chi_1\}) \\ &= \int_{q^{-1}(A)} \prod_{i \in D(\chi_0)} Q_{ie_i}(e_{\chi_0(i)}, dy_i) \prod_{j \notin D(\chi_0)} \epsilon_{e_j}(dy_j) \tau(y_1, \ldots, y_k; \chi_1). \end{aligned}$$

Here $y = (y_1, \ldots, y_k)$ is a variable on E^k.

Let $\kappa_i = y_i e_i^{-1}$; we will view this as a change of variable $\kappa = \alpha_e(y)$ and use it to express the integral as an integral on J^k:

$$\begin{aligned} &\hat{N}(e, \chi_0; q^{-1}(A) \times \{\chi_1\}) \\ &= \int_{\alpha_e(q^{-1}(A))} [\prod_{i \in D(\chi_0)} Q_i(e_{\chi_0(i)} e_i^{-1}, d\kappa_i) \prod_{j \notin D(\chi_0)} \epsilon_\imath(d\kappa_j)] \\ &\qquad \cdot \tau(\kappa_1 e_1, \ldots, \kappa_k e_k; \chi_1). \end{aligned} \tag{2.2}$$

(Recall that $\imath$ is the identity element of J). We need to show that this integral remains the same if e is replaced by e'. Suppose

$$q(e) = q(e') = (\lambda_2, \ldots, \lambda_k);\ \lambda_1 = \imath. \tag{2.3}$$

First, consider the τ term in the integral in Equation 2.2. If $1 \leq i, l \leq k$, then

$$\begin{aligned}(\kappa_i e_i)(\kappa_l e_l)^{-1} &= (\kappa_i \lambda_i e_1)(\kappa_l \lambda_l e_1)^{-1} \\ &= \kappa_i \lambda_i \lambda_l^{-1} \kappa_l^{-1} \\ &= (\kappa_i e_i')(\kappa_l e_l')^{-1}\end{aligned}$$

so that, by the symmetry of the τ-distribution on symmetric frameworks (**7**–2.3),

$$\tau(\kappa_1 e_1, \ldots, \kappa_k e_k; \chi_1) = \tau(\kappa_1 e_1', \ldots, \kappa_k e_k'; \chi_1). \tag{2.4}$$

Secondly, since by 2.3 $e_{\chi_0(i)} e_i^{-1} = \lambda_{\chi_0(i)} \lambda_i^{-1} = e'_{\chi_0(i)} e_i'^{-1}$, we have

$$Q_i(e_{\chi_0(i)} e_i^{-1}, d\kappa_i) = Q_i(e'_{\chi_0(i)} e_i'^{-1}, d\kappa_i). \tag{2.5}$$

Finally, consider $\alpha_e(q^{-1}(A))$. By definition of the change of variables,

$$\begin{aligned}\alpha_e(q^{-1}(A)) &= \{(j_1, \ldots, j_k) \in J^k \,|\, (j_1 e_1, \ldots, j_k e_k) \in q^{-1}(A)\} \\ &= \{(j_1, \ldots, j_k) \in J^k \,|\, q(j_1 e_1, \ldots, j_k e_k) \in A\}.\end{aligned}$$

But by Equation 2.3, $(j_i e_i)(j_1 e_1)^{-1} = j_i \lambda_i j_1^{-1}$, so that

$$q(j_1 e_1, \ldots, j_k e_k) = (j_2 \lambda_2 j_1^{-1}, \ldots, j_k \lambda_k j_1^{-1}) = q(j_1 e_1', \ldots, j_k e_k'),$$

and we get

$$\alpha_e(q^{-1}(A)) = \alpha_{e'}(q^{-1}(A)). \tag{2.6}$$

Putting 2.4, 2.5, and 2.6 together, we see that 2.2 is indeed unchanged upon replacing e with e'. ∎

As a consequence of this theorem and the theorem on respectful descent of chains (Theorem **7**–5.5), we know now that the relativized augmented chain on $J^{k-1} \times \mathcal{I}(k)$ and the relativized standard chain on J^{k-1} are both markovian. Can one expect that the second chain is a descent of the first? In section five we shall see that it is.

3. Diagrammatic representation of descent conditions

In this section we reformulate the notions of respectfulness and decomposability (introduced in chapter seven) in a picturesque manner. We then discuss trace chains and their behavior under descent.

Suppose $(U,\mathcal{U})$ and $(V,\mathcal{V})$ are measurable spaces and $h\colon U \to V$ is a measurable function. As usual, we use the symbol $\mathcal{U}$ (respectively, $\mathcal{V}$) also for the real-valued measurable functions on U (respectively , V).

Recall that if μ is any measure on U, the function h can be used to "push down" μ to a measure on V, called $h_*\mu$:

$$h_*\mu(A) = \mu(h^{-1}(A)), \quad A \in \mathcal{V}. \tag{3.1}$$

If g is any function in $\mathcal{V}$, h can be used to "pull back" g to a function in $\mathcal{U}$, called h^*g:

$$h^*g = g \circ h. \tag{3.2}$$

h^*g is h-measurable (measurable with respect to the subσ-algebra $\{h^{-1}(A) \mid A \in \mathcal{V}\}$ of $\mathcal{U}$); indeed, every h-measurable function arises in this way, i.e., $\mathcal{U} \supset \sigma(h) = h^*\mathcal{V}$.

Now let M be a (positive) kernel on U. In what follows we will find it convenient to think of our kernels in terms of *operators*. Specifically, M is an operator on the function space $\mathcal{U}$: for any $f \in \mathcal{U}$, Mf is the function in $\mathcal{U}$ given by

$$Mf(u) = \int_U M(u,dw)f(w), \quad u \in U. \tag{3.3}$$

Now, if $g \in \mathcal{V}$ then $h^*g \in \mathcal{U}$. Acting on the latter by M we get, as in **7**–4.6, an operator h_*M taking $\mathcal{V}$ into $\mathcal{U}$:

$$[h_*M]g \equiv M(h^*g). \tag{3.4}$$

To see what h_*M looks like as a kernel, choose $g = 1_A$ for $A \in \mathcal{V}$. Then

$$[h_*M](u,A) = M(u,h^{-1}(A)), \quad A \in \mathcal{V}, u \in \mathcal{U}. \tag{3.5}$$

This equation vindicates our use of h_* preceding the M: the symbol h_* acts on the second argument of M just as it does in the usual case of measures on $\mathcal{U}$ as in Equation 3.2.

The above situation is most clearly displayed by means of a commutative diagram:

$$\begin{tikzcd} \mathcal{U} \arrow[r, "M"] \arrow[d, "h^*"'] & \mathcal{U} \\ \mathcal{V} \arrow[ur, "h_*M"] & \end{tikzcd}$$

DIAGRAM 3.6. *Commutative diagram.*

In general, the vertices of a diagram signify objects and the arrows between vertices signify morphisms. A (directed) ***path*** between two objects in a diagram is a sequence of connected arrows from the first object to the second. A diagram ***commutes*** if, for any pair of objects, the composition of morphisms (in order) along any of the paths connecting the objects is the same morphism. The definition of h_*M embodied in Equation 3.4 is the statement that Diagram 3.6 commutes.

We can now display diagrammatically the definition of h-respectfulness of M. Assume that h is bimeasurable and surjective. In Remark **7**–5.3 we pointed out that M is h-respectful iff $h_*M(\cdot, A)$ is constant on fibres of h. But because of the bimeasurability of h, this means that $h_*M(\cdot, A)$ is in $h^*\mathcal{V}$ so that, in fact, M is h-respectful iff $h_*M\colon \mathcal{V} \to h^*\mathcal{V}$. An equivalent way to say this is that M restricts to an operator on $h^*\mathcal{V}$: M, viewed as an operator on the space of $\mathcal{U}$-measurable functions, leaves invariant the subspace $h^*\mathcal{V}$ of h-measurable functions. Now, since $h\colon U \to V$ is surjective, the pullback $h^*\colon \mathcal{V} \to \mathcal{U}$ is injective. Thus, if M restricts to an operator on $h^*\mathcal{V}$, there must be a unique operator, call it R_hM, on $\mathcal{V}$, such that Diagram 3.7 commutes:

In other words, stating the existence of an R_hM (**7**–5.2) that makes Diagram 3.7 commute is equivalent to stating the h-respectfulness of M.

We now turn our attention to h-decomposability (**7**–5.6). Assume M is

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "M"] \arrow[d, "h^*"'] & \mathcal{U} \\
\mathcal{V} \arrow[r, "R_h M"'] \arrow[ur, "h_* M"] & \mathcal{V} \arrow[u, "h^*"']
\end{tikzcd}$$

DIAGRAM 3.7. *h-respectfulness commutative diagram. h is bimeasurable.*

h-decomposable and call m the common version of $m_h^{M(u,\cdot)}$. That m is a kernel says that m is an operator from $\mathcal{U}$ to $\mathcal{V}$. The facts that (i) m is a markovian kernel and (ii) $m(v,\cdot)$ is a measure concentrated on $h^{-1}\{v\}$, for all $v \in V$, are both expressed in saying that (iii) $m \circ h^* = \mathrm{id}_{\mathcal{V}}$ (the identity operator on $\mathcal{V}$). To prove this, note that (i) and (ii) together imply (iii). To get (i) from (iii), apply the latter to the function $1_{V-\{v\}}$. To get (ii) from (iii), apply the latter to the function 1_V. Moreover, to say that m is the rcpd of $M(u,\cdot)$ means that

$$M(u, dw) = \int_V M(u, h^{-1}(dv))m(v, dw).$$

The operator formulation of this is

$$M = h_* M \cdot m \tag{3.8}$$

(where $h_* M$ is as in Equation 3.5). Thus, h-decomposability of M allows us to actually decompose M into a product of kernels (or operators). (Such an operator decomposition is not posited for general M.) Indeed we may state conversely that if there exists a *markovian* kernel m on U relative to V such that Equation 3.8 holds and such that $m \circ h^*$ is the identity on $\mathcal{V}$, then m *must* be the common rcpd of $M(u,\cdot)$ relative to h; a fortiori, M is h-decomposable. We thus have Diagram 3.9:

$$
\begin{tikzcd}
 & \mathcal{U} \arrow[r, "M"] \arrow[d, "m"] & \mathcal{U} \\
\mathcal{V} \arrow[ur, "h^*"] \arrow[r, "id"] & \mathcal{V} \arrow[ur, "h_*M"] &
\end{tikzcd}
$$

DIAGRAM 3.9. *h-decomposability commutative diagram. The left-hand triangle says that m is a markovian kernel with $m(v,\cdot)$ supported on $h^{-1}\{v\}$.*

Now we may also display the kernel D_hM introduced in Definition **7**–5.6. Recall that, as operators,

$$D_hM = m(h_*M). \tag{3.10}$$

This is displayed in Diagram 3.11. The rest of the diagram commutes.

Remark 3.12. The operator $h^*: \mathcal{V} \to \mathcal{U}$ is itself a kernel; explicitly,

$$h^*(u, dv) = \epsilon_{h(u)}(dv).$$

4. Trace chains and their descent

We now turn to the question of trace chains; we will use the terminology of **5**–1.

Assumption 4.1. $\{X_n\}$ is a canonical Markov chain with state space $(U, \mathcal{U})$, base space $(\Omega, \mathcal{G}, M_\nu)$ where $\Omega = U^\infty$, and the "past" σ-algebras are $\mathcal{G}_n =$

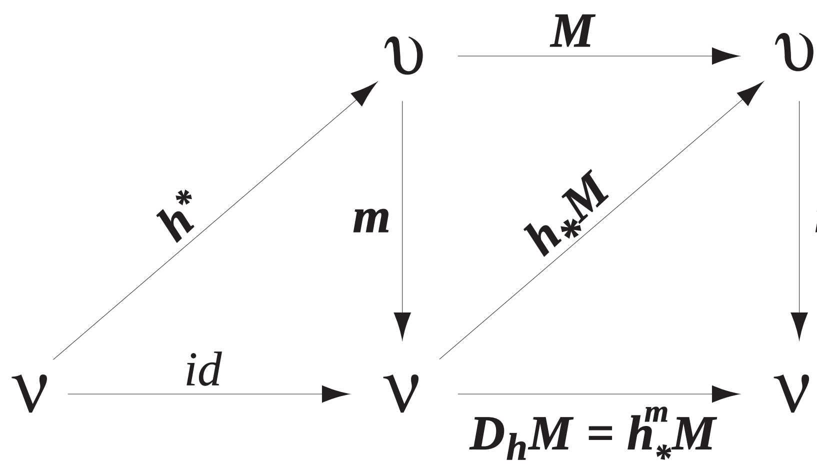


DIAGRAM 3.11. *h-decomposability commutative diagram. m is markovian.*

$\sigma(X_1, \ldots, X_n)$. The one-step transition probabilities of the chain are given by the sequence of kernels $M = \{M_n\}_{n \geq 0}$ on U. ν is a measure on U which is the initial measure of the chain. M_ν is the measure on Ω associated to ν via the sequence of kernels M (in the manner described in **7**–5.9).

Suppose T is some stopping time. Then it may be checked that $T + 1$ is also a stopping time. For each $n \geq 0$, the nth occurrence of T is a stopping time which may be defined in terms of T by the following device: for any stopping time S, let $\theta_S \colon \Omega \to \Omega$ be the random variable given by

$$\theta_S(\omega) = \begin{cases} \theta_n(\omega) & \text{if } S(\omega) = n \\ (\Delta, \Delta, \ldots) & \text{if } S(\omega) = \infty \end{cases}$$

(where Δ is the cemetery). The successive occurrences of T are then the stopping times $\{T_n\}_{n \geq 0}$, defined inductively by

$$\begin{gathered} T_0 = T, \quad T_n = T_{n-1} + T \circ \theta_{T_{n-1}+1}, \\ \mathcal{G}_{T_n} = \{A \in \mathcal{G} \mid A \cap \{T_n = k\} \in \mathcal{G}_k, \forall k \in \mathbf{N}\}, \end{gathered} \tag{4.2}$$

where $\mathcal{G}_{T_n}$ is called the *σ-algebra associated to T_n*. The basic fact here is given by the next theorem:

Theorem 4.3. Define the sequence of random variables $\{Y_n\}_{n\geq 0}$ by $Y_n(\omega) = X_{T_n(\omega)}(\omega)$. *Then Y_n is $\mathcal{G}_{T_n}$-measurable, and the sequence $\{Y_n\}$ is a Markov chain on the base space $(\Omega, \mathcal{G}, M_\nu)$ with respect to the σ-algebras $\mathcal{G}_{T_n}$.*

We are interested here in the following case: Let C be a measurable subset of U, and define T_C, *the first hitting time of C,* and S_C, *the first return time of C,* as follows:

$$\begin{aligned} T_C(\omega) &= inf\{n \geq 0 |\, X_n(\omega) \in C\}, \\ S_C(\omega) &= inf\{n \geq 1 |\, X_n(\omega) \in C\}. \end{aligned} \tag{4.4}$$

T_C and S_C are stopping times. Recalling that I_C is the operator given by $(I_C f)(u) = 1_C(u) f(u)$ for any random variable f on U, the following result is standard in the theory of Markov chains:

Theorem 4.5. Let $C \in \mathcal{U}$ with $\nu(C) > 0$. Let $T = T_C$ and let T_n be as in Equation 4.2. Then the Markov chain $\{Y_n = X_{T_n}\}$ has transition probabilities given by

$$\begin{aligned} &\Pi_n^C(M) \\ &= I_C M_n I_C + I_C \sum_{k\geq 1} (M_n I_{C^c} M_{n+1} I_{C^c} \ldots M_{n+k-1} I_{C^c}) M_{n+k} I_C \end{aligned} \tag{4.6}$$

and initial distribution ν^C given by

$$\begin{aligned} \nu^C(A) &= M_\nu[X_{T_C(\cdot)}(\cdot) \in A] \\ &= (\nu I_C)(A) + \sum_{k\geq 1} (\nu M_0 I_{C^c} M_1 I_{C^c} \ldots M_k I_C)(A), \end{aligned} \tag{4.7}$$

where $A \in \mathcal{G}$.

Definition 4.8. The chain $\{Y_n\}$ in Theorem 4.5 is called the *trace chain on C,* or *the chain induced on C.*

If the original chain $\{X_n\}$ descends via some measurable map $h\colon U \to V$, does the trace chain $\{Y_n\}$ also descend? We first consider respectful descent. We collect some useful facts about products of kernels:

Proposition 4.9. Suppose K, L are kernels on U, and $h\colon U \to V$ is measurable. Then

(i) $h_*(KL) = K(h_*L)$;

(ii) If h is also bimeasurable and if L is h-respectful, $h_*(KL) = (h_*K)(R_hL)$;

(iii) If h is bimeasurable and both K and L are h-respectful, then KL is also h-respectful, and

$$R_h(KL) = R_hK \cdot R_hL.$$

Proof. Consider Diagram 4.10, where the dotted lines constitute the assumptions of parts (ii) and (iii).

$$\begin{tikzcd}
\mathcal{U} \arrow[r, "L"] & \mathcal{U} \arrow[r, "K"] & \mathcal{U} \\
\mathcal{V} \arrow[u, "h^*"] \arrow[ur, "h_*L"] \arrow[urr, "h_*(KL)"] \arrow[r, dashed, "R_hL"'] & \mathcal{V} \arrow[ur, "h_*K"] \arrow[r, dashed, "R_hK"'] & \mathcal{V} \arrow[u, "h^*"']
\end{tikzcd}$$

DIAGRAM 4.10. *Respectful descent.*

The proposition follows from this diagram, in view of the respectfulness criterion of Diagram 3.7. ∎

Theorem 4.11. Suppose $h\colon U \to V$ is bimeasurable. Let $C = h^{-1}(C')$ with $C' \in \mathcal{V}$. Suppose the chain $\{X_n\}$ descends *respectfully* via h to the chain $\{X'_n\}$ on V. Then the trace chain $\{Y_n\}$ of $\{X_n\}$ on C descends respectfully via h to the trace chain $\{Y'_n\}$ of $\{X'_n\}$ on C'.

Moreover, if the transition probabilities of $\{X'_n\}$ are denoted $R_hM = \{R_hM_n\}$, the transition probabilities of $\{Y'_n\}$ are given by

$$\Pi_n^{C'}(R_hM) = R_h(\Pi_n^C(M)).$$

Proof. We need to show that the $\Pi_n^C(M)$ as given in Equation 4.6 are h-respectful, and the equation above holds.

In 4.6, $\Pi_n^C(M)$ is expressed as a sum of terms. To show that it is h-respectful it suffices to show that each summand is h-respectful. Now each of these terms is a product, so that we can use Proposition 4.9. Indeed, the kernels M_n are respectful by hypothesis. The kernels I_C and I_{C^c} are respectful as we observed in **7**–5.4; the formulae presented there show moreover that $R_h(I_C) = I_{h(C)} = I_{C'}$, and similarly $R_h(I_{C^c}) = I_{(C')^c}$. It then follows directly from 4.9, part (iii), that $\Pi_n^C(M)$ is h-respectful, with

$$\begin{aligned} R_h(\Pi_n^C(M)) =& I_{C'}(R_hM_n)I_{C'} \\ &+ I_{C'}\sum_{k\geq 1}(R_hM_n)I_{(C')^c}\dots(R_hM_{n+k-1})I_{(C')^c}(R_hM_{n+k})I_{C'}. \end{aligned}$$

But this is evidently the same as $\Pi_n^{C'}(R_hM)$. This concludes the proof. ∎

We are going to apply Theorem 4.11 so that the role of h is played by the relativization map $\hat{q}$ of section two. We have seen (Theorem 2.1) that $\hat{N}$ is $\hat{q}$-respectful. Thus the augmented position chain $\{\hat{X}_n\}$ on $\hat{E}^k = E^k \times \mathcal{I}(k)$ descends respectfully via $\hat{q}$ to a chain $\{Z_n\}$ on $\hat{J}^{k-1} = J^{k-1} \times \mathcal{I}(k)$, which is called the ***augmented relative position chain;*** it represents the dynamics from the perspective of, say, the first participator. However, to make the representation relevant to a study of that participator's perception, we must consider the chain only at the participator's proper time. This amounts to taking the trace of $\{Z_n\}$ on the subset $C' = J^{k-1} \times C_1$ of $\hat{J}^{k-1}$, where $C_1 = \{\chi \in \mathcal{I}(k) | \, 1 \in D(\chi)\}$ $= \{$those channelings which involve the first participator$\}$. In this section and the next one, however, all of our results are true for an arbitrary subset $\tilde{C}$ of $\mathcal{I}(k)$, not just for C_1. Thus, in general we will let C' denote $J^{k-1} \times \tilde{C}$, and we will write $C = E^k \times \tilde{C}$.

We will need to make explicit the relationship between the trace of $\{Z_n\}$ on C', and the trace of $\{\hat{X}_n\}$ on C. This is done in the next theorem and is depicted in the diagram below.

$$\begin{array}{ccccc}
 & & \text{trace on } C & & \text{augmented chain} \\
 & & \Longleftarrow & & \\
\{\hat{X}_n^C\}, \Pi^C(\hat{N}) & C & = E^k \times \tilde{C} \subset & \hat{E}^k & \{\hat{X}_n\}, \hat{N} \\
\Downarrow & \big\downarrow \hat{q} & & \big\downarrow \hat{q} & \Downarrow \\
\{Z_n^{C'}\}, R_{\hat{q}}(\Pi^C(\hat{N})) & C' & = J^{k-1} \times \tilde{C} \subset & \hat{J}^{k-1} & \{Z_n\}, R_{\hat{q}}(\hat{N}) \\
 & & \Longleftarrow & & \\
 & & \text{trace on } C' & & \text{augmented relative}
\end{array}$$

DIAGRAM 4.12. ***Relationship between trace and relativization.***

Theorem 4.13. Let $\tilde{C}$ be any subset of $\mathcal{I}(k)$. Let $C = E^k \times \tilde{C}$ and $C' = J^{k-1} \times \tilde{C}$. Let $\{\hat{X}_n^C\}$ denote the trace on C of the augmented position chain $\{\hat{X}_n\}$, and let $\{Z_n^{C'}\}$ denote the trace on C' of the augmented relative position chain $\{Z_n\}$. Then $\{\hat{X}_n^C\}$ descends respectfully to $\{Z_n^{C'}\}$ via $\hat{q}$, and the transition probability of $\{Z_n^{C'}\}$ is

$$R_{\hat{q}}(\Pi_n^C(\hat{N})) = \Pi_n^{C'} R_{\hat{q}}(\hat{N}).$$

Proof. This is an immediate corollary of 4.11, noting that $C = \hat{q}^{-1}(C')$. ∎

Let us now turn to decomposable descents.

Proposition 4.14. Let $h{:}\, U \to V$ be measurable. Let C be a measurable subset of U, and ρ a finite positive measure on U. Let ρ_C denote the restriction of ρ to C. Let m be a version of the rcpd of ρ with respect to h. Then
(i) $h_*\rho_C\{v \in V \,|\, m(v, C) = 0\} = 0$, and
(ii) a version of the rcpd of ρ_C with respect to h is given by

$$m_C(v, du) = \frac{1}{m(v, C)} m(v, du) 1_C(u).$$

Proof. First we establish that

$$h_*\rho_C(dv) = m(v, C) h_*\rho(dv). \qquad (*)$$

For, if $D \in \mathcal{V}$,

$$\begin{aligned} h_*\rho_C(D) &= \rho(C \cap h^{-1}(D)) \\ &= \int h_*\rho(dv) \int m(v, du) 1_C(u) 1_D(h(u)), \end{aligned}$$

by the assumed rcpd decomposition (**7**–4.12). $m(v, \cdot)$ is concentrated on $h^{-1}\{v\}$. Thus the integral above is zero if $v \notin D$; otherwise it equals $m(v, C)$. Hence

$$h_*\rho_C(D) = \int h_*\rho(dv) m(v, C) 1_D(v),$$

giving (*).

The conclusion (i) immediately follows upon (*). Because of (i), the kernel m_C in (ii) is well-defined. Furthermore, it is markovian, and $m(v, \cdot)$ is supported on $h^{-1}\{v\}$. That the decomposition

$$\rho_C(A) = \int h_*\rho_C(dv) m_C(v, A), \quad A \in \mathcal{U},$$

holds is now easily checked using (*). ∎

Proposition 4.15. Let K and L be kernels on U, $h: U \to V$ be measurable, and L be h-decomposable with common rcpd m_L. Then KL is h-decomposable, and m_L is also a common rcpd for KL, so that

$$D_h(KL) = m_L K(h_*L).$$

Proof. Consider Diagram 4.16:

The right-hand triangle commutes since $h_*(KL) = Kh_*L$ by part (i) of 4.9. The left and middle triangles commute since L is h-decomposable by the diagrammatic criterion 3.9. In view of this, the result follows by applying 3.11 to KL. ∎

Theorem 4.17. Let $\{X_n\}$ be a Markov chain on U with transition probabilities $\{M_n\}$ and initial distribution ν. Let $C \in \mathcal{U}$. Let $h: U \to V$ be measurable, and suppose that $\{X_n\}$ descends decomposably via h to $\{X'_n\}$ on V. Let $\{Y_n\}$ denote the trace chain of $\{X_n\}$ on C with transition probabilities $\Pi_n^C(M)$ and initial distribution ν^C as in 4.6 and 4.7.

$$\begin{tikzcd}
 & \upsilon \arrow[r, "L"] \arrow[d, "m_L"'] & \upsilon \arrow[r, "K"] & \upsilon \\
\nu \arrow[ur, "h^*"] \arrow[r, "id"'] & \nu \arrow[ur, "h_*L"] \arrow[urr, "h_*(KL)"'] & &
\end{tikzcd}$$

DIAGRAM 4.16. *Commutative diagram.*

Let m denote a common rcpd of M and ν. Put

$$m_C(v, du) = \frac{1}{m(v, C)} m(v, du) 1_C(du).$$

Then the $\Pi_n^C(M)$ and ν^C have common rcpd m_C. It follows from Theorem **7**–5.8 that $\{Y_n\}$ descends decomposably via h to a Markov chain $\{Y'_n = h(Y_n)\}$ on V, with initial distribution $h_*\nu^C$ and transition probabilities

$$D_h(\Pi_n^C(M)) = m_C h_*(\Pi_n^C(M)).$$

Proof. We first consider the kernel $\Pi_n^C(M)$. By 4.6 this is a sum of terms. For simplicity we will temporarily denote the kth summand by P_k, so that $\Pi_n^C(M) = \sum_{k \geq 0} P_k$. Each P_k is itself a product which ends with $M_{n+k} I_C$. By hypothesis these M_{n+k} have the same (common) rcpd m. Now the product $M_{n+k} I_C$ means that the measures $M_{n+k}(u, \cdot)$ are restricted to C. Therefore we can apply 4.14 to deduce that these $M_{n+k} I_C$ have common rcpd m_C. Then by 4.15 it follows that each P_k has the same common rcpd m_C. Thus we can write

$$\begin{aligned}\Pi_n^C(M) &= \sum_k P_k \\ &= \sum_k (h_* P_k) m_C \\ &= (h_* \sum_k P_k) m_C \\ &= h_*(\Pi_n^C(M)) m_C.\end{aligned}$$

This means that the $\Pi_n^C(M)$ have common rcpd m_C as claimed.

It remains to prove the assertions about ν^C. The proof that its rcpd is m_C is almost identical to the proof for $\Pi_n^C(m)$ above: We use the expression 4.7, where ν^C is also written as a sum of products, each ending with $M_{n+k} I_C$. The previous terms of the product must now be viewed as measures starting with ν; however, measures are the special case of those kernels constant in the first argument, so we can again use 4.15 as above. ∎

In contrast to Theorem 4.13 we are not concerned here with the question of whether the descended chain $\{Y_n\}$ is itself a trace chain. In particular we do not assume here that $C = h^{-1}(C')$ for some $C' \in \mathcal{V}$.

We now apply 4.17 to the situation where the map h is $p\colon \hat{E}^k \to E^k$ (**7**–4), and the chain $\{\hat{X}_n\}$ on $\hat{E}^k$ is the augmented position chain. We have seen that the kernels $\hat{N}$ are p-decomposable with rcpd τ, so that $\{\hat{X}_n\}$ descends decomposably via p to the standard chain $\{X_n\}$ on E^k (**7**–4.10 and **7**–5.8). As in Theorem 4.13, we let $C = E^k \times \tilde{C}$, and we take the trace of $\{\hat{X}_n\}$ on C; we will here denote this trace chain by $\{\hat{X}_n^C\}$. $\{\hat{X}_n^C\}$ has transition probabilities given by the kernels $\Pi_n^C(\hat{N})$, and initial distribution μ^C (assuming an initial distribution μ of $\hat{X}_n$). With this notation, we arrive at the next theorem:

Theorem 4.18. Let $\tilde{C}$ be a subset of $\mathcal{I}(k)$, and let $C = E^k \times \tilde{C}$. Let

$$\tau_C(x, d\hat{z}) = \frac{1}{\tau(x, C)} \tau(x, d\hat{z}) 1_C(\hat{z}).$$

Then $\Pi_n^C(\hat{N})$ and μ^C have common rcpd τ_C with respect to p. Consequently, the trace chain $\{\hat{X}_n^C\}$ descends decomposably via p to a chain $\{X_n^C\}$ on E^k.

This latter chain has initial distribution $p_*(\mu^C)$ and transition probabilities

$$D_p(\Pi_n^C(\hat{N})) = \tau_C p_*(\Pi_n^C(\hat{N})).$$

Proof. This is an immediate corollary of 4.17. ∎

The situation is summarized in the diagram below.

$$\begin{array}{cccccc}
 & & & \text{trace on } C & & \text{augmented chain} \\
 & & & \Longleftarrow & & \\
\{\hat{X}_n^C\}, \Pi_n^C(\hat{N}) & E^k \times \tilde{C} = & C & \subset & \hat{E}^k & \{\hat{X}_n\}, \hat{N} \\
\Downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & \Downarrow \\
\{X_n^C\}, D_p(\Pi^C(\hat{N})) & & E^k & = & E^k & \{X_n\}, D_p(\hat{N})
\end{array}$$

DIAGRAM 4.19. *Relationship between standardization and trace on C.*

Remark 4.20. What is the relationship between the chains $\{X_n\}$ and $\{X_n^C\}$ in the diagram? Here $p(C) = E^k$, so C is not the inverse image by p of *any* subset of E^k. Therefore we cannot expect that $\{X_n^C\}$ is itself a trace chain. However, we can describe the situation as follows: As before, let T denote the hitting time of the subset C of $\hat{E}^k$ for the chain $\{\hat{X}_n\}$, so that $\hat{X}_n^C = \hat{X}_{T_n}$. T is not the hitting time, in general, of any subset of E^k for the chain $\{X_n\}$. Nevertheless, T *is a stopping time for* $\{X_n\}$. This happens because, in our case, p is bimeasurable, so that if $p^\infty\colon (\hat{E}^k)^\infty \to (E^k)^\infty$ is the map induced by p, for any $A \in \sigma(\hat{X}_0, \dots, \hat{X}_n)$ we will have $p^\infty(A) \in \sigma(X_0, \dots, X_n)$. Applying this to the sets $A_n = T^{-1}\{n\}$ gives the result. It follows that $X_n^C = X_{T_n}$ is the Markov chain of Theorem 4.3. Notice that this gives another proof that $\hat{X}_n^C$ descends; however, it fails to make explicit the *type* of descent.

5. Compatibility of multiple descents

Theorem 5.1. Let $(\hat{U},\hat{\mathcal{U}})$, $(U,\mathcal{U})$, $(\hat{V},\hat{\mathcal{V}})$, and $(V,\mathcal{V})$ be measurable spaces. Let $\hat{K}$ be a kernel on $\hat{U}$. Let $p\colon \hat{U} \to U$, $r\colon \hat{V} \to V$ be measurable, and let $\hat{q}\colon \hat{U} \to \hat{V}$,

$q: U \to V$ be bimeasurable and surjective. Suppose $r \circ \hat{q} = q \circ p$, so that the following diagram commutes.

$$\begin{tikzcd}
\hat{V} \arrow[d, "r"'] & \hat{U} \arrow[l, "\hat{q}"'] \arrow[d, "p"] & \hat{K} \\
V & U \arrow[l, "q"] &
\end{tikzcd}$$

DIAGRAM 5.2. *Hypotheses of Theorem 5.1.*

Suppose that (1) $\hat{K}$ is $\hat{q}$-respectful, (2) $\hat{K}$ is p-decomposable, and (3) there is a version m of the common rcpd of $\hat{K}$ with respect to p such that, when we view m as an operator $m: \hat{\mathcal{U}} \to \mathcal{U}$, the image of $m \circ \hat{q}^*$ is contained in the image of q^* in $\mathcal{U}$. Then

(i) $R_{\hat{q}}(\hat{K})$ is r-decomposable, with common rcpd n determined uniquely by $m \circ \hat{q}^* = q^* \circ n\,$;

(ii) $D_p(\hat{K})$ is q-respectful; and

(iii) $R_q D_p(\hat{K}) = D_r R_{\hat{q}}(\hat{K})$.

Proof. For simplicity, we denote $\hat{M} = R_{\hat{q}}\hat{K}$, $K = D_p(\hat{K})$, and $M = D_r\hat{M}$. (We need to prove M exists.) We refer to Diagram 5.3. q^* is injective since q is surjective.

In accordance with the premises of the theorem, the solid arrows in 5.3 already constitute a commutative diagram. The ***top face*** displays the $\hat{q}^*$-respectfulness of $\hat{K}$. The ***right-hand face*** displays the p-decomposability of $\hat{K}$. The ***rearmost slanted face*** is induced by Diagram 5.2. The solid arrow part of the left-hand face is the definition of $r_*\hat{M}$. The commutativity of the ***middle slanted face*** follows from the commutativity of these faces.

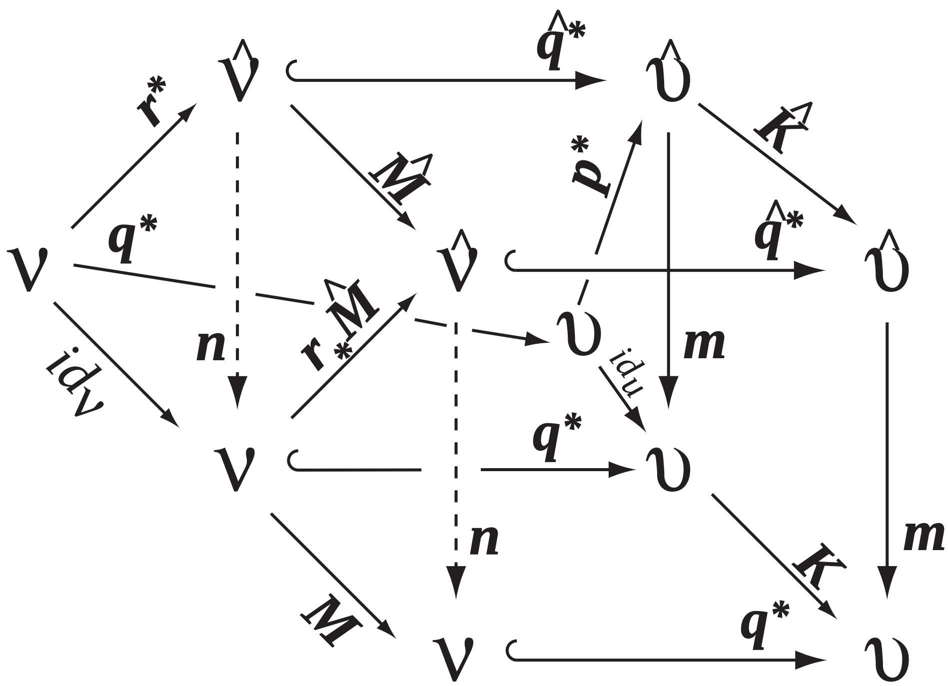


DIAGRAM 5.3. *Commutative diagram.* m *and* n *are markovian.*

The theorem will be proved when we establish the commutativity of the full diagram, including the dotted arrows. For then (i) $\hat{M}$ is r-decomposable by the left-hand face in view of the criterion 3.9 and (ii) K is q-respectful by the bottom face in view of 3.7, whence (iii) $M = D_r\hat{M}$ is then also equal to R_qK.

First, we define n. By hypothesis (3), for $f \in \hat{\mathcal{V}}$ we have $m \circ \hat{q}^*(f) = q^*(g)$ for $g \in \mathcal{V}$; g is unique since q^* is injective. Thus we can define $n(f) = g$. The ***inner vertical face*** then commutes by construction of n. This defines n as an operator, but we need n to correspond to a markovian kernel. Now it is well known that an operator comes from a kernel if and only if it is positive (i.e., preserves positivity of functions) and preserves increasing limits. Thus m, $\hat{q}^*$ and q^* have these properties (recall Remark 3.12); but then so does n, in view of the commutativity relation

$$m \circ \hat{q}^* = q^* \circ n \tag{5.4}$$

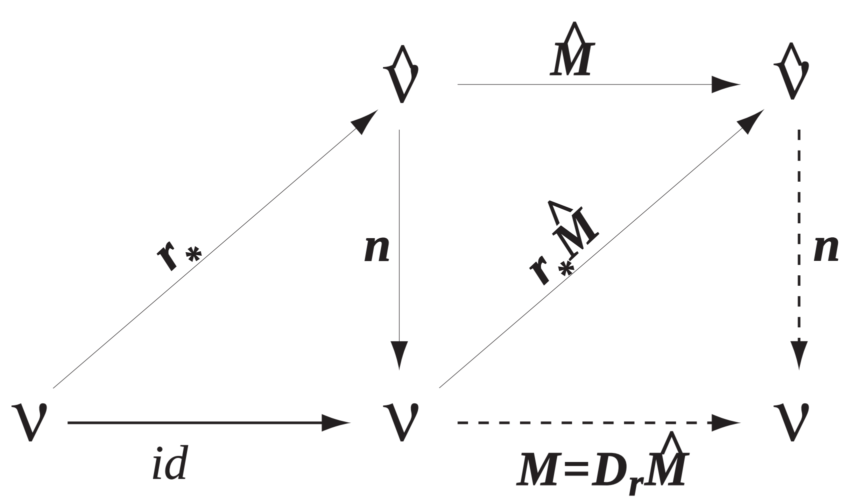
together with the injectivity of $\hat{q}^*$ and q^*. To show that n is markovian, we want $n(1_{\hat{V}}) = 1_V$. Now $r^* 1_V = 1_{\hat{V}}$, so what we want is $n \circ r^*(1_V) = 1_V$. For this, it suffices to show that $n \circ r^* = \mathrm{id}_{\mathcal{V}}$; therefore the markovian property for n will follow from the commutativity of the left-hand face of 5.3.

For the commutativity of this face, we first show that

$$n \circ r^* = \mathrm{id}_{\mathcal{V}}$$
$$\text{and} \qquad \hat{M} = r_* \hat{M} \circ n.$$

Since q^*, $\hat{q}^*$ are injective, it is equivalent to show that (a) $q^* \circ n \circ r^* = q^*$ and (b) $\hat{q}^* \circ \hat{M} = \hat{q}^* \circ r_* \hat{M} \circ n$. By 5.4, the left-hand side of (a) is $m \circ \hat{q}^* \circ r^* = m \circ p^* \circ q^* = \mathrm{id}_{\mathcal{U}} \circ q^*$ (where the equalities depend respectively on the commutativity of the rear slanted face and the rear triangle of the right face). Thus (a) is verified. For (b), by the top face we have $\hat{q}^* \circ \hat{M} = \hat{K} \circ \hat{q}^*$. By the right face $\hat{K} \circ \hat{q}^* = p_* \hat{K} \circ m \circ \hat{q}^*$. By 5.4 this is $p_* \hat{K} \circ q^* \circ n$. Finally by the middle slanted face this is $\hat{q}^* \circ r^* \hat{M} \circ n$.

The current status of the left face is shown in the next unlabeled diagram.



The solid arrow portion of the diagram is now known to commute. But this portion is the decomposability criterion 3.9. It follows from 3.11 that then there exists M so that the whole diagram, including the dotted arrows, commutes.

Thus, the entire left side of 5.3 commutes. The front face commutes since it is a replica of the inner vertical face. Thus the whole diagram is known to commute except the bottom face; but this follows straightforwardly from the commutativity of the other faces. ∎

In applying Theorem 5.1 to the descent of chains, we must be concerned with properties of their initial measures, as well as with properties of their kernels. In particular, for decomposable descent the initial measure must have the same rcpd as the kernel. With the notation as above, suppose then that we have a chain in $\hat{U}$ with initial measure ρ, which descends respectfully via q and decomposably via p. The descent via $\hat{q}$ yields a chain in $\hat{V}$ with initial measure $\hat{q}_*\rho$. To fully exploit 5.1 we will need to know that this measure has the correct rcpd for further decomposable descent via r. The relevant result here is itself a corollary of 5.1.

Corollary 5.5. With the same spaces and functions as in Theorem 5.1, suppose $\hat{\rho}$ is a finite positive measure on $\hat{U}$. Suppose that $\hat{\rho}$ has an rcpd m with respect to p, such that $\mathrm{Im}(m \circ \hat{q}^*) \subset \mathrm{Im} q^*$.

Let $\hat{\sigma} = \hat{q}_*\hat{\rho}$, a measure on $\hat{V}$. Then $\hat{\sigma}$ has an rcpd n with respect to r, uniquely determined by

$$m \circ \hat{q}^* = q^* \circ n. \tag{5.6}$$

Proof. Any positive measure $\hat{\rho}$ may be viewed as a kernel $\hat{K}$ on $\hat{U}$ given by $\hat{K}(\hat{u}, \cdot) = \hat{\rho}(\cdot)$. Since $\hat{K}$ is independent of $\hat{u}$, so is its rcpd m. Thus $\hat{K}$ is automatically p-decomposable. The image of $\hat{K}$ as an operator consists of constant functions: If $f \in \hat{\mathcal{U}}$ then $\hat{K}f(\hat{u}) = \hat{\rho}(f) \equiv \int_{\hat{U}} \hat{\rho}(d\hat{u}) f(\hat{u})$. Moreover, if $A \in \hat{\mathcal{V}}$ then

$$\hat{q}_*\hat{K}(\hat{u}, A) = \hat{q}_*\hat{\rho}(A) = \hat{\sigma}(A).$$

Since this is a constant function in $\hat{\mathcal{U}}$, it is, a fortiori, constant on fibres of $\hat{q}$. Therefore $\hat{K}$ is $\hat{q}$-respectful, and $R_{\hat{q}}(\hat{K}) = \hat{\sigma}$.

Thus conditions (1) and (2) of Theorem 5.1 are satisfied for $\hat{K}$. Condition (3) is also satisfied by hypothesis. We conclude from the theorem that n is uniquely determined by 5.6 and is the rcpd of $R_{\hat{q}}(\hat{K})$, i.e., of $\hat{\sigma}$. ∎

We now apply our results to observer chains in the setting of Hypothesis 2.0. As in section four, we let $\tilde{C}$ be an arbitrary subset of $\mathcal{I}(k)$, $C = E^k \times \tilde{C} \subset \hat{E}^k$, and $C' = J^{k-1} \times \tilde{C} \subset \hat{J}^{k-1}$. We then make the following identifications in Theorem 5.1 and Corollary 5.5:

$$
\begin{aligned}
\hat{U} &= C, \quad U = E^k, \quad \hat{V} = C', \quad V = J^{k-1}, \\
p &= \text{ restriction to } C \text{ of } \mathrm{pr}_1 \colon \hat{E}^k \to E^k, \\
r &= \text{ restriction to } C' \text{ of } \mathrm{pr}_1 \colon \hat{J}^{k-1} \to J^k, \\
q &= \text{ the relativization map as in } \S 2, \\
\hat{q} &= \text{ the restriction to } C \text{ of the } \hat{q} \text{ in } \S 2, \\
\hat{K} &= \Pi_j^C(\hat{N}) \text{ for any } j, \text{ and} \\
\hat{\rho} &= \hat{\mu}^C.
\end{aligned}
\tag{$**$}
$$

As usual, $\hat{N}$ denotes the sequence of kernels of the augmented position chain, and the $\Pi_j^C(\hat{N})$ is as defined in 4.6. $\hat{\mu}$ is the initial measure of the augmented chain, and $\hat{\mu}^C$ is as defined in 4.7. This includes the case where $\tilde{C} = \mathcal{I}(k)$, so that $C = \hat{E}^k$, $C' = \hat{J}^{k-1}$, $\hat{K} = \hat{N}$, etc.

$$
\begin{tikzcd}
J^{k-1} \times \tilde{C} \arrow[d, equal] & E^k \times \tilde{C} \arrow[d, equal] & \\
C' \arrow[d, "\mathrm{pr}_1|_{C'} = r"'] & C \arrow[l, "\hat{q}"'] \arrow[d, "p = \mathrm{pr}_1|_C"] & \Pi_n^C(\hat{N}),\ \mu^C \\
J^{k-1} & E^k \arrow[l, "q"']
\end{tikzcd}
$$

DIAGRAM 5.7. *Commutative diagram.*

We now observe that with the identifications (**), the hypotheses of Theorem 5.1 (and Corollary 5.5 for $\hat{\rho}$) are satisfied. In fact, Diagram 5.2 becomes

Diagram 5.7 which is commutative, q and $\hat{q}$ are bimeasurable and surjective, $\hat{K}$ is $\hat{q}$-respectful (Theorem 4.13) and p-decomposable (Theorem 4.18). For the common rcpd m of $\hat{K}$ we may take τ_C, as given in 4.18.

It remains to check the hypothesis (3) of 5.1: $\mathrm{Im}(\tau_C \circ \hat{q}^*) \subset \mathrm{Im}(q^*)$. This is true by virtue of the configuration symmetry of the τ-distribution (part (3) of Definition **7**–2.3). To see this explicitly, take f to be a measurable function on C'. Then for $e \in E^k$,

$$(\tau_C \circ \hat{q}^* f)(e) = \frac{1}{\tau(e, \tilde{C})} \sum_{\chi \in \tilde{C}} \tau(e; \chi) f(\hat{q}(e, \chi)). \tag{5.8}$$

Now the configuration symmetry of τ means exactly that for any fixed χ the mapping $e \mapsto \tau(e; \chi)$ is constant on the fibres of q; we have already noted this after 2.0. Recalling that $\hat{q}(e, \chi) = (q(e), \chi)$, this implies that the right side of 5.8, viewed as a function of e, is constant on the fibres of q, i.e., it is in the image of q. By Theorem 4.18 the measure μ^C also has rcpd τ_C; therefore, the hypotheses of Corollary 5.5 are also satisfied for $\hat{\rho} = \mu^C$.

Thus, we can apply Theorem 5.1 and Corollary 5.5 to the situation in Diagram 5.7, i.e., to the trace chain on C of the augmented position chain. We get the following theorem, which also summarizes Theorems 4.13 and 4.18:

Theorem 5.9. Let $\{X, Y, E, S, G, J, \pi\}$ be a symmetric framework, with E principal homogeneous for J. Let a configuration symmetric τ be given. Assume we have k participators with symmetric action kernels Q_1, …, Q_k and initial measures $\xi_1, \ldots, \xi_k$. Let $\hat{\mu} = (\xi_1 \otimes \ldots \otimes \xi_k)_\tau$ and $\hat{N}_n = \{\langle Q_1(n), \ldots, Q_k(n)\rangle_\tau\}$ be the initial measure and one-step transition probabilities for the corresponding augmented dynamical chain $\{\hat{X}_n\}$. Let $\{X_n\}$ denote the standard chain, and let $\{\hat{Z}_n\}$ and $\{Z_n\}$ denote the augmented relative chain and standard relative chain respectively. Let $\tilde{C} \subset \mathcal{I}(k)$ be any subset, let $C = E^k \times \tilde{C}$ and $C' = J^{k-1} \times \tilde{C}$. Let $\hat{q}$ and q be the relativization maps of section two, and let $p\colon \hat{E}^k \to E^k$ and $r\colon \hat{J}^{k-1} \to J^{k-1}$ be projections on the first factor.

Consider the Diagram 5.10, in which each double arrow indicates the chain-construction procedure as labelled. The chains in the front face of the diagram with superscript C and C' notation are defined to be the result of the appropriate arrow.

The conclusion: *Diagram 5.10 exists and is commutative.* The commutativity here means that any two sequences of procedures which have the same beginning and the same ending yield the same result. The "stopped chain" terminology means that $X_n^C = X_{T_n}$ where $T = T_C$, and $Z_n^{C'} = Z_{T_n}$ where $T = T_{C'}$. Here the use of T_C and $T_{C'}$ as stopping times is as discussed in Remark 4.20.

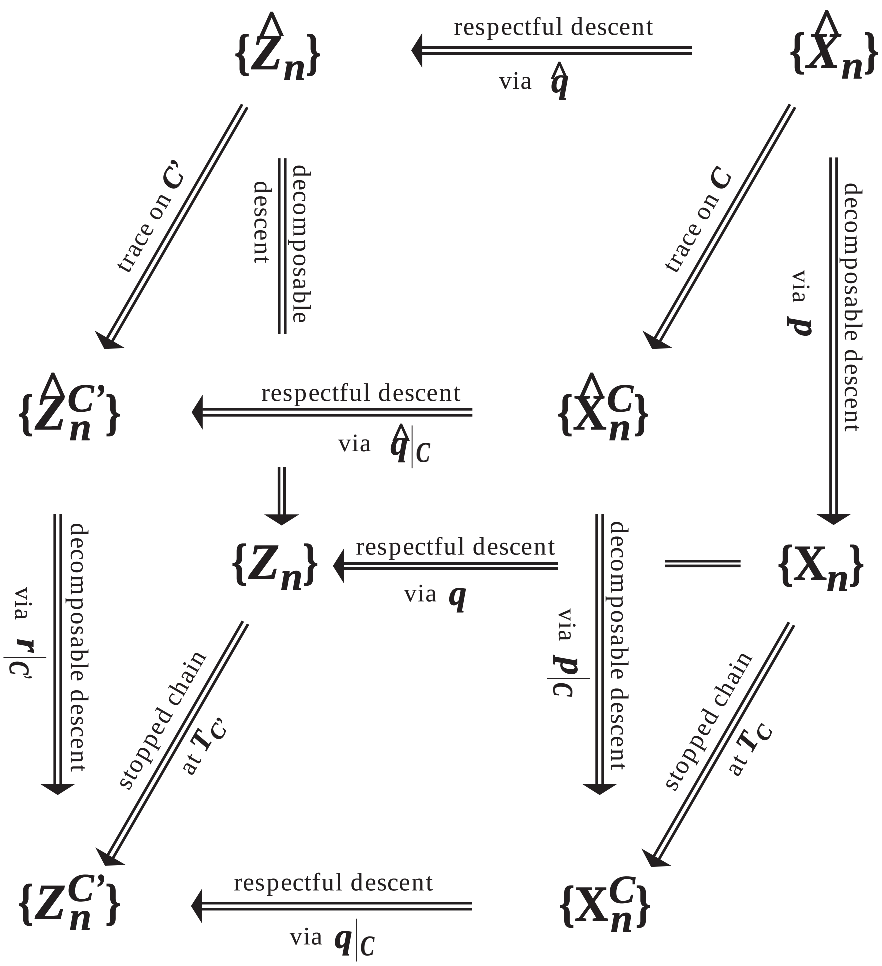


DIAGRAM 5.10. ***Commutative diagram, relating the various dynamical chains.***

Remark 5.11. The commutativity of 5.10 contains the appropriate assertions about the initial distributions of the chains in question. For example, the rcpd with respect to $r|_C$ of $(\hat{q}_*(\hat{\mu}))^{C'}$ is the same as that of $\hat{q}_*(\hat{\mu}^C)$.

6. Matching perception to reality

We assume that we are in a symmetric framework with E principal homogeneous for J, and with a symmetric τ-distribution. In this setting suppose that we have k participators with symmetric action kernels. Thus we will continue to use the notation of 2.0.

It is reasonable to consider the augmented position chain $\{\hat{X}_t\}$ on $\hat{E}^k$ to be the "ultimate source" of phenomena—meaning those phenomena which arise in, or are associated to, the participator dynamics. This point of view is justified by Theorem 5.9: The theorem tells us firstly that the "derived" stochastic processes $\{X_t\}$, $\{\hat{Z}_t\}$, $\{Z_t\}$, $\{\hat{X}_t^C\}$, $\{X_t^C\}$, $\{\hat{Z}_t^{C'}\}$, $\{Z_t^{C'}\}$ are Markov chains on the same base space, which we may take to be the canonical space $\hat{\Omega}$ of the chain $\{\hat{X}_t\}$. Moreover, the theorem affirms that the character of any one of these chains is not an artifact of the particular sequence of descents used to derive it; this character depends only on the way that the given chain is probabilistically grounded in $\hat{\Omega}$. It is in this sense that the probability space $\hat{\Omega}$—which is informationally equivalent to the chain $\{\hat{X}_t\}$—is seen as the common source.

The dictionary[1] defines ***phenomenon*** as "anything directly apprehended by the senses or one of them: an event that may be observed: the appearance which anything makes to our consciousness: ...." One might paraphrase this (with apologies) by saying that ***phenomena are the constituents of a subjective reality.*** With this definition, while $\{\hat{X}_t\}$ may be viewed as the source of phenomena as above, it is not itself ***phenomenal.*** In fact, the derived chains $\{X_t\}$, $\{\hat{Z}_t\}$, $\{Z_t^{C'}\}$, ... (other than $\{\hat{X}_t\}$) are more appropriately called the ***phenomenal chains.***

For example, we will speak of the ***subjective reality chain*** of, say, the first participator. As we noted in section one, the participator is ignorant of the full channeling involution: it is aware only of being channeled to, and the successive instances of this awareness define its proper time. This means that the subjective or phenomenal reality of this participator is already contained in the chain $\{X_t^C\}$ where $C = E^k \times C_1$, $C_1 = \{\chi \in \mathcal{I}(k) |\, 1 \in D(\chi)\}$. Moreover, if we suppose that the participator's interpretation kernel is symmetric (as in **5**–5.6), then its conclusions are actually conclusions about the chain $\{Z_t^{C'}\}$ (where the relativization is, of course, with respect to the same first participator): the participator's subjective reality is contained in $\{Z_t^{C'}\}$. As we will see,

[1] Kirkpatrick, E.M. (editor), ***Chambers 20th Century Dictionary,*** Press Syndicate of the University of Cambridge, New York, 1983.

the relativization procedure imposes a strong form of "unknowability" on the unrelativized chains: the existence of a stationary probability measure (i.e., a stable phenomenology) on a relativized chain does not imply the existence of such a measure for the corresponding absolute chain.

In the study of specialization one considers phenomenal chains defined by subsets $\tilde{C}$ of $\mathcal{I}(k)$ more general than C_1. They correspond to subsystems of our k-participator system which function as a single ("higher level") observer. In any case, we conceptualize the various derived chains as phenomenal or subjective reality chains for suitable participators or specialized subsystems of participators. All of these chains partake of a common probabilistic source which is itself unknowable by the participators: the augmented absolute chain $\{\hat{X}_t\}$. Traditionally, the word ***noumenon*** denotes "an unknown and unknowable substance or thing as it is in itself."[2] Thus we might also call the chain $\{\hat{X}_t\}$ ***the noumenal chain,*** the inaccessible unity underlying the separate possible subjective realities.

We now study in more detail a single participator's view of the dynamical situation. We will assume that the participator's interpretation kernel is symmetric, i.e., that its perception (as well as its action) is relativized. This means that the "view" in question is appropriately expressed by the ***relativized*** chain $Z^{C'} = \{Z_t^{C'}\}$ of 5.9. This chain may be obtained, for example, as the relativization of X^C, or as the chain Z stopped at the time $T_{C'}$, or even as the standardization of $\hat{Z}^{C'}$. Here we use the terminology (and results) of 5.9, and we put

$$\begin{aligned} C_1 &= \{\chi \in \mathcal{I}(k) \,|\, 1 \in D(\chi)\}, \\ C &= E^k \times C_1, \\ C' &= J^{k-1} \times C_1. \end{aligned} \tag{6.1}$$

We have taken the first participator as the distinguished one. The stopping time $T_{C'}$ or T_C is the proper time of the first participator. The relativization $X \Longrightarrow Z$ is taken with respect to the first participator as usual, i.e., it is the respectful descent via q or $\hat{q}$ as in section two.

Terminology 6.2. We will call our distinguished participator ***participator*** A. The chain $Z^{C'}$ will be called A***'s (subjective) reality chain.*** η will denote A's ***fundamental interpretation kernel*** (**5**–5.6). (This notation is for simplicity: a priori, η is not time invariant, and when we wish to note this we will write $\eta(t)$.)

[2] ibid.

What does it mean to say that A's perception is matched to its reality? At each moment of its proper time, a point s lights up in S, and A's interpretation of this is the probability measure $\eta(s, \cdot)$ on J. *This is A's interpretation of the position of the source of the channeling relative to its current position.* Knowledge of A's absolute perspective e would enable the output $\eta_e(s, \cdot)$, which is a probability measure on E, but we are here concerned only with the relativized situation $Z^{C'}$. Thus it is reasonable to make the following preliminary definition.

Definition 6.3. A's perception, as embodied in its interpretation kernel η, matches A's reality at time t if, for any measurable subset K in J, $\eta(t)(s, K)$ is the actual probability (in the chain $Z^{C'}$) that the manifestation of at least one participator has a perspective differing from that of A by an element of the set K, given that the channeling to A at time t results in s.

This definition takes into account the fact that A's subjective reality cannot include the details of the channeling involution, i.e., we are in $Z^{C'}$ and not, say, $\hat{Z}^{C'}$. It follows that the criterion given above is not sensitive to which participator truly channeled to A. Instead, the definition asserts that $\eta(t)(s, K)$ (which is in any case the same as $\eta(t)(s, \pi^{-1}(s) \cap K)$) is the actual probability that the manifestation of a participator, having a perspective differing from that of A by an element of the set K, could have channeled to A at time t, resulting in s, i.e., that $\pi^{-1}(s) \cap K$ was occupied at this time. We now obtain an a priori expression for this probability.[3] This will enable us to express Definition 6.3 in the form of an equation.

Suppose that the distribution of the $(k-1)$-dimensional random vector $Z_t^{C'}$ is ν_t; this is a distribution on J^{k-1}. Then the inclusion-exclusion principle allows us to compute the probability that at least one of the k-participators lies in K. The procedure is formalized in the following definition:

Definition 6.4. To each measure ζ on J^{k-1} we associate a measure on J,

[3] Since we condition on a *value* of s, we should be using the expression "regular conditional probability distribution" rather than just "probability." This will be taken as understood in what follows.

denoted $\mathcal{D}\zeta$, as follows: Let $K \in \mathcal{J}$. For $1 \leq i \leq k-1$, let

$$K_i = \left(\prod_{l=1}^{i-1} J \right) \times K \times \left(\prod_{l=i+1}^{k-1} J \right)$$

(i.e., K_i is the cartesian product of $k-2$ copies of J with one copy of K in the ith place). Then let

$$K_{i_1, i_2, \ldots, i_l} = \left(\bigcap_{j=1}^{l} K_{i_j} \right)$$

for $1 \leq i_1 < i_2 < \ldots < i_l \leq k-1$. Then put

$$\mathcal{D}\zeta(K) = \sum_{l=1}^{k-1} (-1)^{l+1} \sum_{1 \leq i_1 < \ldots < i_l \leq k-1} \zeta(K_{i_1, \ldots i_l}).$$

The assignment $\zeta \mapsto \mathcal{D}\zeta$ is linear, and $\mathcal{D}\zeta$ is a probability measure if ζ is one.

In consequence of this definition we have the following proposition.

Proposition 6.5. Let ζ be a probability measure on J^{k-1}. Then

$$\begin{aligned} &\zeta\{(v_1, \ldots, v_{k-1}) \in J^{k-1} | \text{ at least one of the } v_1, \ldots, v_k \text{ lies in } K\} \\ &\qquad = \mathcal{D}\zeta(K). \end{aligned}$$

Proof. The set in question is just $\bigcup_{i=1}^{k-1} K_i$, so the result is a direct application of the inclusion-exclusion principle and the definition of $\mathcal{D}\zeta$. ∎

In particular, if ν_t is the distribution of $Z_t^{C'}$, $(\mathcal{D}\nu_t)(K)$ is the probability that at least one component lies in K. It follows:

Proposition 6.6. Let ν_t denote the distribution of $Z_t^{C'}$. Then the conditional probability that the manifestation of at least one participator has a perspective differing from that of A by an element of the set $K \subset J$ at time t, given that the channeling to A results in s, is

$$m_\pi^{\mathcal{D}\nu_t}(s, K)$$

(rcpd notation as in **7**–4.12).

We can now make our preliminary definition precise.

Definition 6.7. We will say that "A's perception matches its reality at time t," or that "A has true perception at time t" if

$$\eta(t) \text{ is a version of the rcpd } m_{\pi}^{\mathcal{D}\nu_t}.$$

If this holds *for all t,* we will simply say that "A's perception matches its reality," or "A has true perception."

The words "true" and "perception," like the word "reality," are technical terms in the above definition. "Reality" is the subjective reality chain $Z^{C'}$. In keeping with our probabilistic semantics, reality at time t is the ***distribution*** ν_t of the state $Z_t^{C'}$, and not the discrete states themselves. The word "perception" denotes that which is representable by the interpretation kernel η; in particular A's perceptual representations are made in the framework J (and not J^{k-1}). Thus $\mathcal{D}\nu_t$ is that aspect of the reality ν_t which is perceptually representable. Perception is "true" if the representation $\eta(t)$ agrees with this representable aspect of reality modulo the observer structure embodied by the map π. Perceptual truth is therefore several semantic levels removed even from the "subjective reality" $Z^{C'}$. This in turn is several levels removed from the "source" or "noumenal" reality $\hat{X}$. (And from the standpoint of the whole lattice of observer families, $\hat{X}$ itself is a localization.)

The time-dependent, or instantaneous character of the definition of true perception given in Definition 6.7 is required for semantic completeness: The interpretation kernel η is, a priori, time-dependent. The action kernels of the participators are time-dependent; in every respect the participator is a dynamical entity. Even if all the action kernels were time-independent, so that the chain $Z^{C'}$ is homogeneous, the distribution ν_t will in general depend on t, and hence so will $\mathcal{D}\nu_t$. However, from both the intuitive and analytic viewpoints and for purposes of both application and theoretical development, the fundamental situation occurs when $Z^{C'}$ has a stationary measure. This "stable reality" context gives rise to an important modification of Definition 6.7.

Recall that a measure ν is ***stationary*** for a Markov chain $\{\xi_t\}$ if $\nu P_t = \nu$ for each one-step transition probability P_t of the chain; νP denotes the operation of the kernel P on ν defined in **6**–1. It is equivalent to say that if the distribution of ξ_0 is ν, then for all $t \geq 0$ the distribution of ξ_t is also ν. Stationary measures do not always exist; when they do, they are not necessarily unique. In general, if ν_t denotes the distribution of ξ_t, the ν_t, though not stationary themselves, may converge to a stationary measure as $t \to \infty$.

Definition 6.8. We say that A has ***stably true perception*** if $\eta(t)$ is independent of t, and is a version of $m_\pi^{\mathcal{D}\nu}$ for a fixed stationary measure ν of $Z^{C'}$. More generally, we say that A has ***stably true perception in the limit*** or that A ***tends*** (or ***converges***) ***to stably true perception*** if $lim_{t\to\infty}\,\eta(t)$ is a version of $m_\pi^{\mathcal{D}\nu}$ for a stationary ν.

In order to maintain flexibility, no hypothesis is made in the definition about the relationship between the actual distribution ν_t of the chain and the stationary measure ν. (The presumption, however, is that either $\nu_t = \nu$ for all t or $\nu_t \to \nu$ as $t \to \infty$.) Nor has any particular form of convergence been specified.

How good is stably true perception? Let us assume that for each t, $\nu_t = \nu$, a stationary measure. Then ***the participator A with stably true perception instantiates, at each instant of its proper time, an observer whose inferences are inductively strong.*** Indeed, this is the observer $(G,\, Y,\, J,\, S,\, \pi,\, \eta)$ whose event set is J, whose perspective map π is the same as the fundamental map π of our original symmetric framework (X, Y, E, S, G, J, π), and whose conclusion kernel is A's fundamental interpretation kernel—the one which satisfies Definition 6.8. In fact, by hypothesis the measure $\mathcal{D}\nu$ correctly (and time-invariantly) describes the distribution in J of the population of A's "universe." This is the universe consisting of participators in the original dynamical ensemble, but only insofar as they channel with A. The conclusion kernel η then correctly describes this population distribution, conditioned by the element $s \in S$ resulting from channeling; this is the very meaning of inductive strength of an observer inference. Note further that if we imagine A to have some kind of "access" to the distribution $\pi_*(\mathcal{D}\nu)$ on S, then A knows the actual distribution $\mathcal{D}\nu$, not just its π-rcpd.

Can A make any inductively strong inference beyond that of inferring the location of anonymous channelers to A? In the first place, A has no means of identifying the other participators as individuals or even of inferring the number of participators in the ensemble. Thus, there is no basis for inferring ν itself even if $\mathcal{D}\nu$ is known. Of course, we can imagine that A builds a representation consisting of one other participator whose relative position has time-invariant distribution $\mathcal{D}\nu$, and then we might argue that this is a strong inference. However, it is really just a canonical form for the same inference as before. For example, there is no inference here of the actual number of participators. In attempting to infer from relative to absolute position, an even more fundamental obstruction arises. For here it is possible that relative positions have a stationary (probability) distribution while the absolute positions do not. We can get an example of this by considering a two participator system involving A and B; assume that the position of B relative to A has a stationary distribution while A itself executes, say, a transient random walk. These considerations and others suggest that stably true perception per se does not lead *canonically* to inductively strong inferences at a higher level than that of the subjective reality of the observer (G, Y, J, S, π, η).