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Estimating Item and Order Information

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In a common psychological procedure, the subject is presented a sequence of items and is asked to recall them in order. His response is scored for items reported correctly in their correct position (position score) and for items reported correctly independently of position (item score). Such data are analyzed in terms of a model which assumes that a particular stimulus item may be forgotten entirely (State 0), may be remembered but without any knowledge of its position in the sequence, or may be remembered together with knowledge of its position (State 2). State 2 is related to the position score, and we define a nonexclusive State 1 (which contains all items not in State 0) that is related to the item score. In Part 1, we use the observed item and position scores to derive estimates of the trial-to-trial distribution of the number of items in States 1 and 2. In Part 2 we consider separately each serial position of the stimulus, and derive estimates of the probability that each individual item is in State 1 and State 2. The model handles omissions, second guesses, and gives sensitive estimates of partial information. Fast Fortran computer programs are available for all computations. In general, whenever responses are scored for items and/or for position, and when no alternative model is being tested, it is recommended that the above model be used to correct for the effects of guessing.

In a common psychological procedure, the subject is presented an ordered set of items (for example, nine spoken numerals in an immediate memory test) and asked to recall them in order. He is required to produce a requisite number of response items, guessing whenever he is uncertain. The subject can make two kinds of errors: He can make an *item* error, that is, he responds with an item that did not occur in the stimulus; and, he can make an *order* error, that is, he recalls an item that did occur in the stimulus but his recall is in the wrong order. We assume throughout that stimulus items are chosen *without* replacement.

We propose to analyze data from such experiments in terms of a model that assumes, for each stimulus item, one of three mutually exclusive events occurs: The particular item may be forgotten entirely, or it may be remembered without any knowledge of where it occurred in the stimulus sequence, or it may be remembered together with knowledge of its position (order) in the stimulus. For convenience, we define three *non*exclusive states as follows. State 2 contains the perfectly remembered items; i.e., those that have retained "item" and "order" information. State 1 contains all the items in State 2 plus the additional remembered items that have lost "order" information. State 0 contains the forgotten items. (The reason for defining States this way is to make the number of items in State 1 correspond to the "true item score" and the number of items in State 2 correspond to the "true position score"—see below.)

We apply the model to experiments consisting of a number of trials, on each of which a new stimulus is presented and a response is recorded. In Part 1 of this paper, we derive estimates of the probability Y(y) that on a particular trial exactly y items are in State 2 and the probability X(x) that exactly x items are in State 1. We make these estimates from just two statistics accumulated on each trial:

(1) the number of response items written correctly and in their correct position (the observed *position* score, v),

(2) the number of response items that "match" stimulus items (the observed *item* score, u). More precisely, the observed item score is the position score that would be recorded if, before scoring, the order of the response items was permuted so as to maximize the position score. Figure 1 shows an example of how these scores are obtained.

We will demonstrate an exact correspondence between the observed item and position scores on the one hand and the number of items in States 1 and 2 on the other (Fig. 2). The treatment offered here is not a statistical reduction of the data; it is a transformation of the data to eliminate the effects of chance guessing. There are exactly as many parameters in the transformed data as in the original data. We shall

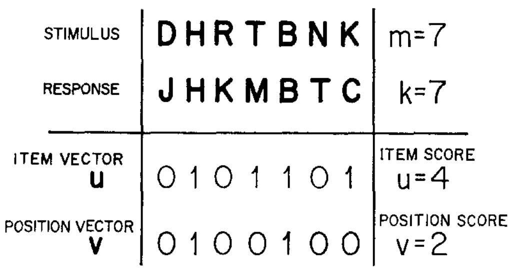

FIG. 1. A typical stimulus and response (top) and the response vectors derived therefrom (bottom). A "1" in a particular position indicates that the corresponding stimulus item is reported correctly. The sequence of zeros and ones is regarded as a vector; the number of ones is the score. See text for details.

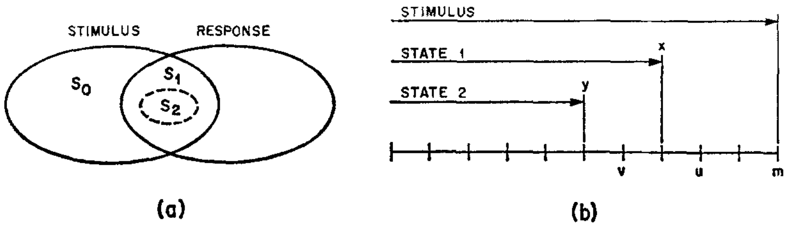

FIG. 2. (a) The fate of stimulus items. Some remain is State 0, the rest are in State 1 (item knowledge); some of those in State 1 are also in State 2 (item-plus-position knowledge). (b) The effect of guessing on the outcome of a single trial. The number of stimulus items is *m*. The item score is u (resulting from x items in State 1 and guessing); the position score is v (resulting from y items in State 2, x items in State 1, and guessing).

use the letters u, v, respectively, to refer to observed item and position scores, and the letters x, y to refer to the corresponding transformed quantities, that is, to the number of items in States 1 and 2, respectively, of the model. Capital letters refer to probability distributions: we shall use U, V, W to refer to probability distributions of item scores, of position scores, and the joint distribution of item and position scores, respectively; and the letters X, Y, Z to refer to the corresponding states in the model, that is, to the probability distribution of items in State 1, in State 2, and the joint distribution of items in State 1 and State 2, respectively.

The calculation of the distribution of items in States 1 and 2 is conceptually simple; it merely requires the computation of how much the subject can improve his performance by intelligent guessing, given the various kinds of information that he has. For example, when the subject remembers (in State 2) eight of nine presented items and their positions, and he also remembers (in State 1) a ninth item without any knowledge of its position, it is assumed he will produce the ninth item in its correct position because only one response position remains unfilled. Similarly, if he does not remember the ninth item, but he knows that the stimulus is composed by sampling without replacement from a set of nine alternatives, then it is assumed that the eight items he knows will enable him to deduce the ninth. In general, guessing computations are much more complex, but because the algebraic formulas can be stated recursively, the resulting expressions can be evaluated by relatively simple computer programs.

From the distribution functions of items in States 1 and 2, one can trivially compute their means, $\bar{x}$, $\bar{y}$; that is, "the mean true number of items remembered without regard to position" (the item span) and "the mean true number of items remembered with knowledge of position" (the order span), two spans that so often have been incorrectly computed in the literature that citations are unnecessary here.

The Need for a Model to Analyze Data

The advantage of having a model to analyze the data is that guessing—particularly in the case of item scores—is a major factor in observed scores. Consider, for example, an extreme case for a finite alphabet. In the limit as stimulus and response size gets as large as the alphabet, the expected item score approaches 100% of the maximum score if the subject merely writes down all the characters of the alphabet. For example, in a digit span task, a subject is read nine digits and asked to recall them. Assuming there are no repeats in the stimulus sequence, the subject guarantees an average item score of 8.9 (of a possible 9.0) merely by writing nine different randomly selected digits as his response. Only when the subject's item score exceeds 8.9 can we infer any item knowledge whatsoever.

In a more typical case involving a memory experiment, Sperling and Speelman (1970, pp. 169–170) observed, "in almost every task, data for AD (acoustically different) and AS (acoustically similar) stimuli differ less from each other in their item score than in their position score." In terms of the model we can ask: Does using acoustically similar (rather than normal) stimuli reduce the number of items in State 2 more than in State 1; i.e., is there a selective deleterious effect of acoustic similarity on the retention of order information? By using Δu as the change in score between the acousticallu similar and the acoustically different conditions, we can phrase the problem more succinctly: Does the experimental observation of $\Delta u > \Delta v$ imply $\Delta x > \Delta y$? Sperling and Speelman, in fact, showed that $\Delta x \approx \Delta y$; i.e., that acoustic similarity did not have a selective effect on order information, and that the differences in scores ($\Delta u > \Delta v$) could be accounted for by the effects of guessing.

Arguments in favor of using an all-or-none model of information about an item (as opposed to a continuous state model) have been proposed by Sperling and Speelman (1970). (In fact, the present model is a three-state model.) Basically, the rationale is: (1) There are fewer parameters in an all-or-one model; (2) the all-or-none model is typical of the mid-range of continuous models; and (3) the simplicity of the all-or-none model makes it convenient to express results as being equivalent to a certain all-or-none process even when performance is generated by a continuous process.

1. Scores Summed over All Serial Positions

Number of Items in State 1

In the first part of this paper we are only concerned with the total number of items in State 1 and in State 2. We refer to the total number of items in State 1 as the subject's "true item score" or as the number of items the subject "knows."

The observed item score is related to the underlying ***true*** item score through a guessing matrix, G. G(x; u) is the conditional probability that, when the subject knows exactly x items, he will get a total of u items correct ($u \geqslant x$) by guessing exactly u—x items correctly (Fig. 2b).

The Guessing Matrix, G

The stimulus is composed of m different items, chosen randomly without replacement, from an alphabet of l different items. The response consists of k different items. (The numbers m, l, and k are fixed for a particular experiment and $l \geqslant \max((k, m).)$ For the cases in which $k \neq m$, it is useful to define $m' = \max(m, k)$ and $k' = \min(m, k)$. Note that when $k \leqslant m$, $m' = m$ and $k' = k$. (The cases in which $k \neq m$ are discussed in the section *Ambiguity, Omissions, Partial Knowledge, and Second Guesses.*)

The set of l alphabet items can be divided into the set of m items in the stimulus and the set of $l - m$ items not in the stimulus. Furthermore, the set of m stimulus items can be divided into the set of x items, known by the subject to have been presented, and the set of m — x items not known (see Fig. 3).

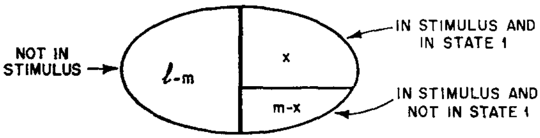

FIG. 3. The fate of the items in the alphabet on a particular trial.

The subject knows x items and it is assumed he guesses k — x different items from the remaining l — x items. G is the probability of choosing (1) u — x items (the number guessed correctly) from the subset of m — x items presented but not known by the subject and (2) k — u items (the number guessed wrongly) from the subset of $l - m$ items not presented. So,

$$G(x; u) = \binom{m - \mathrm{x}}{\mathrm{u} - \mathrm{x}}\binom{l - m}{k - \mathrm{u}} \Big/ \binom{l - \mathrm{x}}{k - \mathrm{x}}, \tag{1}$$

where the binomial coefficient $\binom{a}{b}$ is

$$\binom{a}{b} = \frac{a!}{(a-b)!\,b!}, \qquad a \geqslant b,$$

$$\binom{a}{b} = 0, \qquad a < b.$$

Figure 4 illustrates G(x; u) for the case of a 6-item stimulus drawn from a 17-item alphabet.

True Item Score

The relationship between the ***true*** item score and the ***observed*** item score in terms of the guessing matrix just defined is

$$\mathrm{U(u)} = \sum_{\mathrm{x}=0}^{\mathrm{u}} \mathrm{G(x; u)\, X(x)}, \tag{2}$$

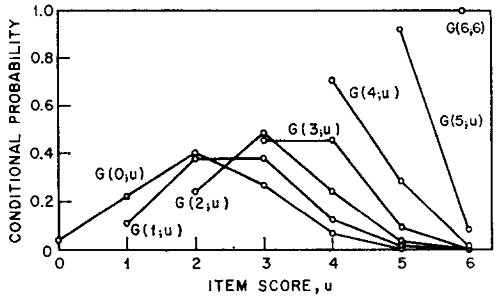

FIG. 4. Conditional probability of observing an item score of exactly u items given that exactly x items are in State 1, where x = 0, 1, 2, 3, 4, 5, 6 for the various curves. This example is a graphical representation of the guessing matrix for the case of a 6-item stimulus ($m = 6$) and response ($k = 6$) drawn from a 17-item alphabet ($l = 17$).

where U(u) is the probability of getting u items correct and X(x) is the true probability that the subject knows x items. The summation in Equation (2) represents the contribution to the subject's score of every possible number of known items combined with every possible number of guessed items. We use the observed proportion of responses with exactly u correct items $\hat{U}(u)$ as the estimate of U(u).

The set of equations (2) is a set of $k' + 1$ linear equations with $k' + 1$ unknowns; that is, there is a linear equation (2) for u = 0, 1,..., k'. One can solve this set of equations for X(x) in terms of the U's by matrix methods. However, there is a more instructive and more efficient method of solution. The X's can be solved for iteratively, since to solve for X(n) only X(0) through $X(n - 1)$ are needed. For example, the only way a subject can obtain a score of zero items correct is to know zero items (x = 0) and to guess exactly zero items in k guesses [G(0; 0)]. The probability of a trial with a score of zero is U(0) = G(0; 0)X(0), and we use $\hat{U}(0)$ to solve for X(0). To solve for X(1), we need to know the probability that the subject knows zero items and guesses exactly one item correctly [X(0)G(0; 1)] and the probability that he knows one item and guesses exactly zero items correctly [X(1)G(1; 1)]. In general,[1,2]

$$X(x) = \left(U(x) - \sum_{j=0}^{x-1} G(j; x)\, X(j)\right) \Big/ G(x, x). \tag{3}$$

Equation (3) gives the solution for the probability X(x) of exactly x items in State 1 in terms of the expected scores U(0) ··· U(x). Given any realizable distribution of ob-

[1] The user should not be alarmed at negative probability estimates. When estimating a true probability of zero, we expect an equal number of positive and negative estimates. Occasional negative estimates of probability are intrinsic to unbiased estimation. They occur with all probabilities estimated by this model.

[2] When G in the denominator is zero, it is an ambiguous case. See section on Ambiguity in Item Knowledge.

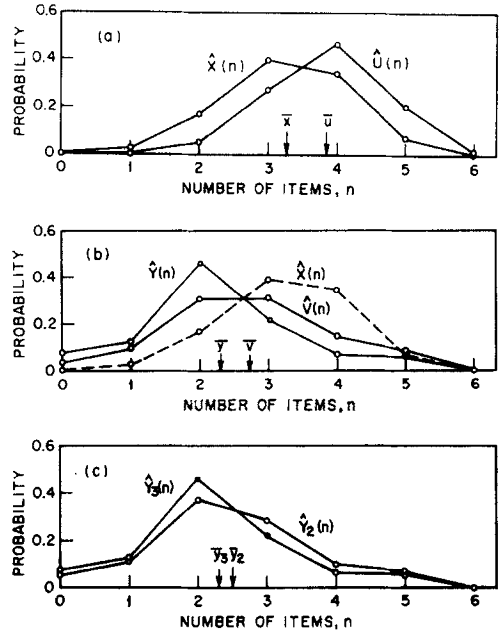

FIG. 5. (a) "Observed" and "true" item scores. Data from a tachistoscopic recognition experiment showing the observed probability $\hat{X}(n)$ of an item score of exactly n items. Six items were presented ($m = 6$) from an alphabet of 17 items ($l = 17$); responses contained six items ($k = 6$); there were 358 trials. The estimated distribution of items in State 1 is $\hat{X}(n)$; $\bar{x} = 3.27$, $\bar{u} = 3.85$. (b) The same data analyzed in terms of a three-state model. The observed probability of a position score of exactly n items correct is $\hat{V}(n)$; the estimated distribution of items in State 2 (position-plus-item knowledge) is $\hat{Y}(n)$. The joint distribution $\hat{W}(u, v)$ is needed to calculate $\hat{Y}(n)$. (c) Estimated distributions of items with position-plus-item knowledge compared for the three-state ($\hat{Y}_3$) and two-state ($\hat{Y}_2$) models. The $\hat{Y}_3$ distribution is the same as in (b).

served item scores $\hat{U}$, the distribution of true item scores $\hat{X}$ that would have generated $\hat{U}$ by random guessing can be calculated from Eq. (3). Figure 5a shows an example of the application of Eq. (3) to some data from an experiment.

Number of Items in State 2

To estimate the number of items in State 1, we used only the observed item scores, U. To estimate the number of items in State 2, we need the joint distribution of item and position scores, W, a much larger data base. This is because the particular item score that goes with a particular position score makes a difference. Thus, in a three-state model, we have to consider all possible combinations of item scores with position scores.

We shall refer to the number of items in State 2 as the subject's "true position score" or as the number of items the subject "knows" together with knowledge of their positions. As in the case of item scores, the observed position score is related to the underlying true position score through a guessing matrix, G′. This position guessing matrix is more complicated than the item guessing matrix because we must consider the subject's item knowledge (number of items in State 1) in computing his position knowledge (number of items in State 2). We consider the general case in which a subject knows x items and of these, he knows y items-and-their-positions. He improves his score by guessing, so that we observe an item score of u and a position score of v. We wish to compute a guessing matrix, G′(x, y; u, v), the conditional probability that if the subject *knows* x items and y items-and-their-positions, we will observe an item score of u because the subject correctly guesses u − x items, and we will observe a position score of v because, by chance, he writes v − y of the u − x items in their correct position.

The Guessing Matrix, G′(x, y; u, v)

The subject can improve his position score by two kinds of guessing. First, he can—by chance—write down in their correct position items he knows without knowledge of their position (i.e., an item in State 1 may appear to be in State 2). Second, he can correctly guess an item that is in State 0 and write it—by chance— in the correct position (i.e., an item 0 may appear to be in State 2).

To calculate the guessing matrix for position scores, we need to calculate the probability that v − y items (the number of positions guessed correctly) out of u − y items (the number of correct items without position knowledge) will be placed in their correct positions when m' − y positions remain to be filled in the response.

Let $N(b, c)$ be the number of ways of placing b different items into c different positions ($b \leqslant c$). So,

$$N(b, c) = c!/(c - b)!.$$

Let $N'(b, c, d)$ be the number of different ways of placing b different items into c different positions ($b \leqslant c$) with a score of exactly d correct placements ($d \leqslant b$). N′ can be defined recursively as follows:

$$N'(b, c, d) = \binom{b}{d}\left[N(b - d, c - d) - \sum_{j=1}^{b-d} N'(b - d, c - d, j)\right]$$

where $\binom{b}{d}$, the binomial coefficient of b and d, is the number of ways of selecting d correct items from b available items and the term in brackets is the number of ways of placing the remaining $b - d$ items, minus the number of ways which have one or more correct items in the remainder. Since we assume the b items are all different, obviously $N'(b, c, b) = 1$, and this is sufficient to solve recursively for all $N'(b, c, d)$.

The probability $R(b, c, d)$ that exactly d items are in their correct position when b items (without position knowledge) are known and/or guessed correctly, and c positions remain to be filled is

$$R(b, c, d) = N'(b, c, d)/N(b, c). \tag{4}$$

Now the guessing matrix for joint item-and-position scores may be written

$$G'(x, y; u, v) = G(x; u)\, R(u - y, m' - y, v - y), \tag{5}$$

where $G(x; u)$ is the quantity defined in Eq. (1) and R is defined in Eq. (4). Equation (5) can be stated in words as follows. When x items are in State 1 and y items are in State 2, the probability $G'(x, y; u, v)$ of observing an item score of u and a position score of v is equal to the probability $G(x, y)$ that the subject improves his item scores by guessing $u - x$ items times the probability that, given u, he improves his position score from y to v by writing $v - y$ of the $u - y$ items in their correct positions.

True Position Scores

Let $W(u, v)$ be the joint probability that the item score is u and the position score is v.

Let $Z(x, y)$ be the joint probability that x items are in State 1 and y items are in State 2.

To derive $Z(x, y)$ from $W(u, v)$, we sum over all the possible ways of obtaining $W(u, v)$:

$$W(u, v) = \sum_{y=0}^{v} \sum_{x=y}^{u} G'(x, y; u, v)\, Z(x, y) \tag{6}$$

We have a set of $\frac{1}{2}(k' + 1)(k' + 2)$ linear equations, one for each pair of u and v such that $u \geqslant v$ ($u = 0, 1, 2, \ldots, k'$; $v = 0, 1, 2, \ldots, u$).

The set of Eqs. (6) can be solved by matrix methods for the Z in terms of the observed W and the coefficients G'. However, when the stimulus size is sufficient to tax the human memory span, this becomes an unwieldy set of equations. For example, when $k' = 7$ there are 36 equations in 36 unknowns, and a 36×36 matrix of coefficients, G'. Therefore, we derive a simple recursive solution analogous to the solutions Eqs. (3) of Eqs. (2).

Using Eq. (5) and separating the summation in Eq. (6) gives

$$W(u, v) = \sum_{y=0}^{v} \left\{ R(u - y, m' - y, v - y) \sum_{x=y}^{u} [G(x; u)\, Z(x, y)] \right\}$$

$$= \sum_{y=0}^{v-1} \left\{ R(u - y, m' - y, v - y) \sum_{x=y}^{u} [G(x; u)\, Z(x, y)] \right\}$$

$$+ R(u - v, m' - v, 0) \sum_{x=v}^{u-1} [G(x; u)\, Z(x, v)]$$

$$+ R(u - v, m' - v, 0)\, G(u; u)\, Z(u, v)$$

Finally, we can derive

$$Z(u, v) = \frac{\left\{ \begin{array}{c} W(u, v) - \sum_{y=0}^{v-1} \left[R(u - y, m' - y, v - y) \sum_{x=y}^{u} [G(x; u)\, Z(x, y)] \right] \\ - R(u - v, m' - v, 0) \sum_{x=v}^{u-1} [G(x; u)\, Z(x, v)] \end{array} \right\}}{R(u - v, m' - v, 0)\, G(u; u)}. \tag{7}$$

The Z(u, v) are computed from the W(u, v) and the Z(x, y), where each pair (x, y) is such that $x \leqslant u$ and $y \leqslant v$ and either $y \leqslant v$ or $x \leqslant u$. This enables Z(u, v) to be computed recursively.[3]

The true position score Y is the marginal distribution of Z(x, y) over all x:

$$Y(y) = \sum_{x=y}^{k'} Z(x, y),$$

where Y(y) is the true probability that the subject knows y items and their positions. Figure 5b shows an example of Y(y) calculated from experimentally observed data (W(u, v)).

We can also write the true item score, X (solved for previously), as the marginal distribution of Z(x, y) over all y:

$$X(x) = \sum_{y=0}^{x} Z(x, y).$$

However, the item scores obtained from marginal distributions of item and position scores are identical to the item scores obtained by considering item scores alone in a two-state model. The model is three-state for position knowledge but only two-state for item knowledge.

[3] Whenever the denominator is zero, the numerator is also zero. When R in the denominator is zero, the quotient is taken as zero. See section on Ambiguity in Position Knowledge. When G in the denominator is zero, then quotient is undefined, and the user must supply a value for the true probability. See section on Ambiguity in Item Knowledge.

A Two-State Model for Position Scores

If, whenever an item is remembered, it is remembered with knowledge of where it occurred, then a two-state rather than a three-state model is appropriate. That is, a particular stimulus item may (1) be remembered with exact knowledge of its position, or (2) be forgotten entirely. In such a case the number of known items x always equals the number of known positions y. Analogously to the simple two-state model for items, observed scores v are generated by knowing y items-and-their-positions and guessing the remaining v − y:

$$V(v) = \sum_{y=0}^{v} G_p(y; v)\, Y(y),$$

where

$$G_p(y; v) = \sum_{u=v}^{k'} G(y; u)\, R(u - y, m' - y, v - y)$$

is the probability that if y items are in State 2 (items with position knowledge), i.e., x = y, then a position score of v will be observed.

Again, solving iteratively yields

$$Y(y) = \left(V(y) - \sum_{j=0}^{y-1} G_p(j, y)\, Y(j)\right) \Big/ G_p(y, y). \tag{8}$$

Equation (8) uses the guessing matrix $G_p(y; v)$ to solve for Y(y) in terms of V(v). An equivalent solution for Z(y, y) in terms of G′(x, y; u, v) and W(u, v) is obtained by solving Eq. (6) with the restriction that x = y. In the two-state position model, Y(y) = Z(y, y).

The two-state position model is applicable when the members of States 1 and 2 are assumed to be identical. Then X(n) and Y(n) become estimates of the same quantity; they merely are based on two different ways of scoring the same data. Figure 5c illustrates the difference between the two- and three-state estimates, $\hat{Y}_1$, $\hat{Y}_2$, for a set of real data in which we expect a priori that $\bar{x} > \bar{y}$, and $\bar{y}_2 \neq \bar{y}_3$.

Procedure for Analysis in Terms of the Three-State Model

(1) Collect the item and position score for each presentation (u, v) to obtain W(u, v).

(2) Computation from Eq. (7) yields Z(x, y) from W(u, v) and the guessing model.

(3) Sum over y to obtain the distribution of items in State 1 (or solve for X(x) separately):

$$X(x) = Z(x, \cdot) = \sum_{y} Z(x, y).$$

Sum over x to obtain the distribution of items in State 2:

$$\mathrm{Y(y)} = \mathrm{Z}(\cdot, \mathrm{y}) = \sum_{\mathrm{x}} \mathrm{Z(x, y)}.$$

(4) The item memory span is E[X(x)]. The position memory span is E[(Y(y)].

2. True Scores for Each Individual Serial Position

In the first part of this paper we calculated the total number of items in State 1 (the true item score) and the total number of items in State 2 (the true position score). The totals were summations over all serial positions. Now the analysis is expanded to take individual serial positions into account; for each individual stimulus position we estimate the probability that an item is in State 1 and the probability that an item is in State 2. The following method of analyzing scores in individual serial positions is analogous to the earlier method of analyzing overall scores. We preserve the earlier notation except that we now use *italic* letters to denote the corresponding distributions in the serial analysis.

Let $X^{(i)}$ be the probability that the ith item of the stimulus is in State 1 (the subject knows the item, with or without knowledge of its position in the stimulus), and let $Y^{(i)}$ be the probability that the ith item of the stimulus is in State 2 (the subject knows the item together with knowledge of its position in the stimulus). The corresponding probabilities of scores are $U^{(i)}$, the probability that the response contains the ith item of the stimulus in any position of the response, and $V^{(i)}$, the probability that the response contains the ith item of the stimulus in the ith position of the response. In general, $U^{(i)} \geqslant X^{(i)}$ and $V^{(i)} \geqslant Y^{(i)}$ because the subject can improve his score by guessing items he does not know.

True Item Scores for Each Serial Position

First we consider only item scores. On any given trial, the response can be represented by a "response vector" $\mathbf{u} = (u_1, u_2, ..., u_m)$ where each u_i is either a 1 or a 0 to represent, respectively, the conditions that the response does or does not contain the item from the ith position of the stimulus (see bottom of Fig. 1). When there are m stimulus items, there are 2^m possible response vectors. For example, when $m = 7$, $\mathbf{u} = (0, 1, 0, 1, 1, 0, 1)$ represents the response of correctly indentifying the items (i.e., they occur in some response position) which occurred in the second, fourth, fifth and seventh positions of the stimulus ($u_2 = u_4 = u_5 = u_7 = 1$) and of being wrong in the remaining positions ($u_1 = u_3 = u_6 = 9$). This response vector is illustrated in line three of Fig. 1.

Let $U(\mathbf{u})$ be the probability of observing the particular response $\mathbf{u}$. Then $U^{(i)}$, the probability that the response contains the ith item of the stimulus is the sum over all responses $\mathbf{u}$ with $u_i = 1$.

$$U^{(i)} = \sum_{\mathbf{u} \ni u_i = 1} U(\mathbf{u}) = \sum_{\mathbf{u}} U(\mathbf{u}) \cdot u_i .$$

Likewise, let $X(\mathbf{x})$ be the probability that the subject "knows" the response $\mathbf{x}$; that is, the fraction of the total number of trials on which the subject knows those items corresponding to the 1's in $\mathbf{x}$ and does not know the items corresponding to the 0's. Then $X^{(i)}$, the probability that the subject knows the ith item of the stimulus, can be written as

$$X^{(i)} = \sum_{\mathbf{x}} X(\mathbf{x}) \cdot x_i . \tag{9}$$

The data consist of the $U(\mathbf{u})$. The problem is to calculate the $X(\mathbf{x})$. Just as X(x) was related to the U(u) in the earlier analysis, the $X(\mathbf{x})$ are related to the $U(\mathbf{u})$ through a guessing matrix, G.

Before calculating the effects of guessing, it will be useful to define a relation of superiority ($\gtrsim$) between any two response vectors, say $\mathbf{a} = (a_1, a_2, ..., a_m)$ and $\mathbf{e} = (e_1, e_2, ..., e_m)$. We say that $\mathbf{a}$ is *superior* to $\mathbf{e}$ ($\mathbf{a} \gtrsim \mathbf{e}$) if and only if $a_i \geqslant e_i$ for all i. Equivalently, we say $\mathbf{e}$ is inferior to $\mathbf{a}$ ($\mathbf{e} \lesssim \mathbf{a}$). Also we say that $\mathbf{a}$ is *strictly superior* to $\mathbf{e}$ ($\mathbf{a} > \mathbf{e}$) or $\mathbf{e}$ is *strictly inferior* to $\mathbf{a}$ ($\mathbf{e} < \mathbf{a}$) when $\mathbf{a}$ is superior to $\mathbf{e}$ but $\mathbf{a}$ is not equal to $\mathbf{e}$. Since guessing can only improve scores, then, when $\mathbf{x}$ is known, only superior $\mathbf{u}$ can be observed; that is, $\mathbf{u} \gtrsim \mathbf{x}$.

The Guessing Matrix, G

Let $G(\mathbf{x}; \mathbf{u})$ be the conditional probability that when the subject knows $\mathbf{x}$, the response $\mathbf{u}$ will be observed. The G matrix is similar to the G matrix defined previously, except now the position of the stimulus items reported in the response is being considered. As before, the stimulus is composed of m different items, chosen randomly without replacement, from an alphabet of l different items, and the response consists of k different items. For any vector $\mathbf{e}$, let e be the number of 1's in $\mathbf{e}$. Then $G(\mathbf{x}; \mathbf{u})$ is the probability of choosing k − u items (the number guessed wrongly) from the subset of $l - m$ items not presented. So,

$$G(\mathbf{x}; \mathbf{u}) = \frac{\mathrm{G(x; u)}}{\binom{m - \mathrm{x}}{\mathrm{u} - \mathrm{x}}} = \frac{\binom{l - m}{k - \mathrm{u}}}{\binom{l - \mathrm{x}}{k - \mathrm{x}}}, \qquad \text{when } \mathbf{u} \gtrsim \mathbf{x} \tag{10}$$

and

$$G(\mathbf{x}; \mathbf{u}) = 0 \qquad \text{when } \mathbf{x} > \mathbf{u}.$$

The numerator of the middle term of Eq. (10) is the probability of improving the score by u − x; that is, the old guessing term, Eq. (1). However, when the score is improved by u − x, **u** is produced only a fraction of the time—the denominator in the middle term—because there may be other **e** such that e = u. Alternatively, the third term of Eq. (10) may be interpreted. The denominator represents the number of ways of making $k - \text{x}$ guesses from among $l - \text{x}$ different items. The numerator represents the number of these guesses that yield u; there is only one way to place the correct items and there are $\binom{l-m}{k-\text{u}}$ ways to place the wrong ones.

Estimating True Item Scores for Each Serial Position

In terms of the guessing matrix $G(\mathbf{x}; \mathbf{u})$, the relationship between the probability of observing a particular outcome $U(\mathbf{u})$ and the underlying probabilities of knowing the responses $X(\mathbf{x})$ is

$$U(\mathbf{u}) = \sum_{\mathbf{x} \lesssim \mathbf{u}} G(\mathbf{x}; \mathbf{u})\, X(\mathbf{x}). \tag{11}$$

The sum is over all **x** inferior to **u** because guessing can only improve the score. Equation (11) is of the same form as Eq. (2). By methods analogous to those developed in Part 1, Eq. (11) can be solved iteratively[2] to give $X(\mathbf{x})$:

$$X(\mathbf{x}) = \Big(U(\mathbf{x}) - \sum_{\mathbf{e} < \mathbf{x}} G(\mathbf{e}; \mathbf{x})\, X(\mathbf{e})\Big)\Big/ G(\mathbf{x}; \mathbf{x}). \tag{12}$$

The $X(\mathbf{x})$ of Eq. (12) can be substituted into Eq. (9) to give the estimated probability (in terms of observed scores) that the subject knows the ith item of the stimulus.

True Position Scores for Each Serial Position

To estimate position scores as well as item scores, a response on any given trial must be represented by two vectors. The "item response vector" **u** was defined previously. We need to define a "position response vector" $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ where v_i is either a 1 or a 0 to represent respectively the conditions that the response does or does not contain the item from the ith position of the stimulus in the ith position of the response (see Fig. 1). When there are m stimulus items, it is possible to have 2^m different **v**'s and 2^m different **u**'s.

Again, as in Part 1, in order to solve for the true position scores in each serial position we first have to solve for the joint distribution of true item and position scores in each serial position. This is because, analogous to the situation in Part 1, the particular position response vector observed on a given trial depends on the item response vector for that given trial.

Let $W(\mathbf{u}, \mathbf{v})$ be the joint probability of occurrence of the item response vector **u** and the position response vector $\mathbf{v}$. Since $u_i \geqslant v_i$, $W(\mathbf{u}, \mathbf{v})$ is only defined for $\mathbf{u} \gtrsim \mathbf{v}$.

Let $Z(\mathbf{x}, \mathbf{y})$ be the joint probability that the subject knows the items represented by the vector $\mathbf{x}$ and the items and positions represented by the vector $\mathbf{y}$. The 1's in $\mathbf{x}$ indicate the stimulus items in State 1, and the 1's in $\mathbf{y}$ indicate the stimulus items in State 2. Since State 2 is a subset of State 1, $Z(\mathbf{x}, \mathbf{y})$ is only defined for $\mathbf{x} \gtrsim \mathbf{y}$.

The data consist of the $W(\mathbf{u}, \mathbf{v})$. The problem is to calculate $Z(\mathbf{x}, \mathbf{y})$. By methods analogous to those of Part 1, we will relate $W(\mathbf{u}, \mathbf{v})$ to $Z(\mathbf{x}, \mathbf{y})$ through a guessing matrix G'.

The Guessing Matrix, $G'(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v})$

Let $G'(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v})$ be the conditional probability that, if the subject knows the item response $\mathbf{x}$ and the position response $\mathbf{y}$, we will observe the item response $\mathbf{u}$ and the position response $\mathbf{v}$. As was explained previously, the subject can improve his position score by two kinds of guessing: (1) he can know an item and by chance place it in the correct response position; or (2) he can guess an item and by chance place it in the correct response position. By analogy to Eq. (5),

$$G'(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v}) = G(\mathbf{x}; \mathbf{u})\, R(\mathrm{u} - \mathrm{y}, m' - \mathrm{y}, \mathrm{v} - \mathrm{y}), \tag{13}$$

where $G(\mathbf{x}; \mathbf{u})$ is given by Eq. (10) and

$$R(\mathrm{u} - \mathrm{y}, m' - \mathrm{y}, \mathrm{v} - \mathrm{y}) = \mathrm{R}(\mathrm{u} - \mathrm{y}, m' - \mathrm{y}, \mathrm{v} - \mathrm{y}) \Big/ \binom{\mathrm{u} - \mathrm{y}}{\mathrm{v} - \mathrm{y}}, \tag{14}$$

where $\mathrm{R}(\mathrm{u} - \mathrm{y}, m' - \mathrm{y}, \mathrm{v} - \mathrm{y})$ is defined by (4). $\mathrm{R}(\mathrm{u} - \mathrm{y}, m' - \mathrm{y}, \mathrm{v} - \mathrm{y})$ is divided by $\binom{\mathrm{u}-\mathrm{y}}{\mathrm{v}-\mathrm{y}}$ because now there is only one way to select $\mathrm{v} - \mathrm{y}$ correct items from $\mathrm{u} - \mathrm{y}$ available items when the serial position (and not merely the total number) of correct items is taken into account.

Estimating True Position Scores for Each Serial Position

By summing over all possible ways of obtaining $(\mathbf{u}, \mathbf{v})$ we find the joint probability $W(\mathbf{u}, \mathbf{v})$,

$$W(\mathbf{u}, \mathbf{v}) = \sum_{\mathbf{y} \lesssim \mathbf{v}} \sum_{\substack{\mathbf{x} \lesssim \mathbf{u} \\ \mathbf{x} \gtrsim \mathbf{y}}} G'(\mathbf{x}, \mathbf{y}; \mathbf{u}, \mathbf{v})\, Z(\mathbf{x}, \mathbf{y}).$$

The conditions that $\mathbf{y} \lesssim \mathbf{v}$ and that $\mathbf{x} \lesssim \mathbf{u}$ reflect the fact that guessing can only improve a response and the condition that $\mathbf{x} \gtrsim \mathbf{y}$ is because State 2 is a subset of State 1. [Note the similarity with Eq. (6).]

Once again the equations can be solved iteratively to give[3]

$$Z(\mathbf{u},\mathbf{v}) = \frac{\left\{ \begin{array}{l} W(\mathbf{u},\mathbf{v}) - \sum\limits_{\mathbf{y}<\mathbf{v}} \left[R(\mathrm{u}-\mathrm{y}, m'-\mathrm{y}, \mathrm{v}-\mathrm{y}) \sum\limits_{\substack{\mathbf{x}\lesssim\mathbf{u} \\ \mathbf{x}\gtrsim\mathbf{y}}} [G(\mathbf{x};\mathbf{u})\, Z(\mathbf{x},\mathbf{y})] \right] \\ \qquad - R(\mathrm{u}-\mathrm{v}, m'-\mathrm{v}, 0) \sum\limits_{\substack{\mathbf{x}<\mathbf{u} \\ \mathbf{x}\gtrsim\mathbf{y}}} [G(\mathbf{x};\mathbf{u})\, Z(\mathbf{x},\mathbf{v})] \end{array} \right\}}{R(\mathrm{u}-\mathrm{v}, m'-\mathrm{v}, 0)\, G(\mathbf{u},\mathbf{u})}. \tag{15}$$

The probability $Y(\mathbf{y})$ that the subject knows the position response $\mathbf{y}$ is then just the marginal distribution of $Z(\mathbf{x}, \mathbf{y})$ over all possible $\mathbf{x}$:

$$Y(\mathbf{y}) = \sum_{\mathbf{x}\gtrsim\mathbf{y}} Z(\mathbf{x},\mathbf{y}).$$

And then $Y^{(i)}$, the probability that the ith item of the stimulus is in State 2 is given by

$$Y^{(i)} = \sum_{y_i=1} Y(\mathbf{y}) = \sum_{\mathbf{y}} Y(\mathbf{y}) \cdot y_i\,.$$

We can also write the $X(\mathbf{x})$, previously calculated by Eq. (12), as the marginal distribution of $Z(\mathbf{x}, \mathbf{y})$ over all $\mathbf{y}$:

$$X(\mathbf{x}) = \sum_{\mathbf{y}\lesssim\mathbf{x}} Z(\mathbf{x},\mathbf{y}).$$

When $Z(\mathbf{x}, \mathbf{y})$ given by Eq. (15) is used to calculate $X(\mathbf{x})$ the results are identical to the previous method. The theory is again three-State for position scores and two-State for item scores.

A Two-State Model for Position Scores for Each Serial Position

In the three-state model the equations for item and position *vectors* (each serial position) are exactly analogous to those for item and position *scores* (summed over all positions). For example, Eq. (7) becomes formally identical to Eq. (15) when the scalar quantities of Eq. (7) are replaced by their corresponding vectors. Similarly, by replacing scalars with their corresponding vectors, the remarks about the scalar two-state model of Part 1 are directly transposable to a vector model for each serial position. So, we can write directly (by analogy to Eq. (8)):

$$Y(\mathbf{y}) = \Big(V(\mathbf{y}) - \sum_{\mathbf{e}<\mathbf{y}} G_p(\mathbf{e},\mathbf{y})\, Y(\mathbf{e})\Big)\Big/ G_p(\mathbf{y},\mathbf{y}), \tag{16}$$

where

$$G_p(\mathbf{y},\mathbf{v}) = \sum_{\mathbf{u}\gtrsim\mathbf{v}} G(\mathbf{y};\mathbf{u})\, R(\mathrm{u}-\mathrm{y}, m'-\mathrm{y}, \mathrm{v}-\mathrm{y}).$$

Ambiguity, Omissions, Partial Knowledge, and Second Guesses

Ambiguity in Position Knowledge

An ambiguity in true position scores occurs when the stimulus and response are of length m, the subject knows the identity and position of $m - 1$ of the m items, and he knows the identity of the mth item. He always writes the mth item in its correct position because only one position remains vacant. In this case, *two* different internal states are indistinguishable. In the corresponding case of vectors, there are $m + 1$ indistinguishable states: the subject can fail to know the position of any single one of m different items and still achieve the same perfect position score as when he knows them all.

When there is no basis for allocating the probability among the m states of imperfect knowledge, Eqs. (7) and (15) with the provision of footnote 3, allocate all the probability to the state of perfect knowledge. The user can, at his discretion, reallocate that probability arbitrarily among the various states of imperfect knowledge. However, it is better to avoid position-score ambiguity by choosing the lengths of the stimulus and response to be more than one item greater than the true position score.

Ambiguity in Item Knowledge

An ambiguity in true item scores occurs when the number of characters in the alphabet l is less than the number of stimulus items k plus response items m; that is, $l < k + m$. Here, the subject cannot avoid writing at least $k + m - l$ items correctly because the number of response items requested (k) is that much greater than the number of possible wrong responses ($l - m$). To analyze the data, the user first must supply values for $X(i)$, $i < k + m - l$ if he is doing an item analysis, and for all $Z(i, j)$, $i < k + m - l$, if he is doing a joint item-and-position analysis. Analogous considerations apply to the vector analyses.

For an analysis of item scores, one way to estimate X(0), X(1),... is to reanalyze subsets of the data from which a sufficient number of items from stimuli and/or responses have been deleted so that the ambiguity vanishes. The subset value $X_S(0)$ is an upper bound for the whole-set value because the opportunity to make more responses generally raises true item scores, that is, moves the histogram of X(x) versus x to the right (see also *Partial Knowledge and Second Guesses*, below). The subset estimates $X_S(0)$, $X_S(1)$,... are used to resolve the ambiguity in X(0), X(1),... . In the special case where a two-state model applies, item ambiguity can be avoided by using position rather than item scores and analyses. However, it is far better to avoid this kind of item ambiguity by using a large-enough alphabet.

Omissions

Omissions refers to the case in which the number of response items, k, is less than the number of stimulus items, m. The model requires that the subject be unconstrained as to which particular positions he is allowed to omit on any trial. Unreported items are assumed to be unknown. It should be obvious that the maximum true score estimated by the analysis cannot exceed the number of items reported. Therefore, the subject must be required to report a number of items at least as large as the largest true score the experimenter hopes to observe, and as often as he expects to observe it. This sets limits on the number of omissions. For the scalar analysis of scores, all responses of the same length are analyzed together. (For the analysis of vectors, all vectors with the same positions omitted are analyzed together.) The X or Y distributions for responses of length 1, 2,..., k ultimately are combined to give the net X and Y distributions.

In terms of the all-or-none model, allowing omissions and analyzing the data separately makes good sense. If the subject knows nothing about an item, guessing can only add variability to the data. When the subject has partial information about items or positions, separate analyses also make good psychological sense; it is likely that trials on which the subject omits an item are the outcome of a different distribution of internal states than are normal trials.

The simplest alternative to omissions is to prohibit them. A more complicated alternative procedure requires the subject to state the confidence of each item in the response. If the items in any confidence category do not improve the response, then this class of responses can be omitted *a posteriori*. (The data are analyzed separately, both with and without the questionable category of responses in order to determine whether or not the category improves the scores. See the section on *Partial Knowledge and Second Guesses* for an analogous procedure.) Although they require more computation, procedures that tolerate omissions yield more powerful estimates of X and of Y than procedures that do not because less guessing means less sampling error.

Partial Knowledge and Second Guesses

Second guesses refers to the case where the subject makes more responses than there are stimulus items; i.e., $k > m$. The analysis assumes that a second guess does not affect the position score but that it can improve the item score. In terms of a strict all-or-none model, second guesses (like all other guesses) give no additional information. However, if the subject has partial information about an item (or, if he is not certain exactly which items he does know) then second guesses can be quite revealing.

We consider the treatment of one kind of partial knowledge of a single item, first without and then with second guesses. (For a different treatment of partial knowledge, in which it is specified as a continuous variable—bits of information—see Sperling and Speelman, 1970, Appendix 1.) We suppose that a subject knows four relevant "features" of an item i whereas five features are needed for identification. This kind of partial knowledge is, strictly speaking, a violation of the assumption of all-or-none item knowledge, according to which the subject can know only all five features or zero features.

Suppose that with knowledge of four or five features the probability of a correct response is $\frac{1}{3}$. The model's estimate of the probability $X^{(i)}$ that the subject "knows" the item is $X^{(i)} < \frac{1}{3}$. The probability $X^{(i)}$ is less than $\frac{1}{3}$ because the model "subtracts" the component that would have resulted from random guessing. Perhaps this $X^{(i)}$ is the very probability the experimenter desires. On the other hand, if the experimenter wishes to characterize the degree of knowledge either as $\frac{1}{3}$ or as the fraction of relevant features known to the subject ($\frac{4}{5}$), then the model yields an undesired number. This kind of violation of the assumption of all-or-none knowledge is not serious, it merely determines the scale for measuring partial knowledge.

To glimpse the potential of permitting second guesses when the subject has partial knowledge, we suppose again that the subject knows four or five relevant features. Suppose that this information restricts his response to three alternatives, each of which has the *a posteriori* probability p of being correct ($p = \frac{1}{3}$). Now suppose we allow the subject to make two "second guesses" in addition to his normal response. As he can now name all three alternatives, this second-guessing procedure guarantees a correct response. The model assesses no penalty for incorrect second guesses (other than correction for their purely random probability of being correct). Therefore, when a subject has sufficient partial information to eliminate any single wrong item, this partial information can be translated into perfect observed and true scores given a large-enough number of second guesses.

To gain more insight, the user can consider the response position of the various second guesses. The model is indifferent to response position in analyzing item scores. However, the subject and experimenter need not be. For example, the experimenter can analyze the same data several times; in the first analysis he omits (ignores) all second guesses; in the second analysis he omits all second guesses except the first one, in the third analysis he omits all second guesses except the first two, etc. In this way the user can determine the incremental contribution (to knowledge of an item) of each additional second guess.

In general, when the subject has partial information, second guesses offer a powerful means to amplify and analyze the partial information. The drawbacks of second guesses are that they complicate the procedure, add variability to the data, and increase the difficulty of interpreting the analysis. It should be noted that even when second guesses are not explicitly requested, they tend to occur when subjects (1) have partial information about some items and (2) have much less information about a remaining item or items, and (3) do not make omissions. In these cases, responses to the remaining items frequently are second guesses to earlier ones.

Avoidance of position ambiguity requires that the length of the stimulus and of the response exceed the true position score by more than one item. Avoidance of implicit second guesses requires that the lengths of the stimulus and of the response exceed the true item score by as little as possible. These constraints in fact express the traditional practice of using stimuli that are a little—but not too much—longer than those that could be reported perfectly.

Computational Considerations

Small Numbers of Observations

Small numbers can be a problem, especially in the vector analyses. For example, with a seven-item stimulus, there are $2^7 = 128$ different item vectors $\mathbf{x}$ and 2187 different pairs of item-and-position vectors $(\mathbf{x}, \mathbf{y})$. With 500 trials in the item vector analysis, this would yield an average of only about four trials per item vector. Obviously, it will not be possible to obtain accurate estimates of all $X(\mathbf{x})$, the true probabilities of occurrence of vectors. Another complication is that some $X(\mathbf{x})$ are much smaller than 1/500. Although a particular $\mathbf{x}$ may be exceedingly rare, some rare $\mathbf{x}$ will occur significantly often if there are many rare $\mathbf{x}$. The occurrence of a rare $\mathbf{x}$ may cause an extremely strange outcome of the vector analysis for that experiment, such as, for example, many $\mathbf{x}$ with estimated $X(\mathbf{x}) << 0$. The problems of too little data can be treated in two ways: (1) discarding certain observations, and (2) grouping the data by categories.

For example, suppose that for a particular condition the probability of a response with zero correct items $U(\mathbf{x}_0)$ is 10^{-4}. We assume here that the subject has no item knowledge whatsoever; any item knowledge would make this probability even smaller. Suppose an experiment consists of 500 trials. Then we expect at most one in twenty experiments to contain a response of $\mathbf{x}_0$. The vector analysis of these one-in-twenty experiments will be grossly abberant because the data contains a vector $\mathbf{x}_0$ whose observed frequency $U(\mathbf{x}_0)$ is at least twenty times greater than its expected frequency. If there are many vectors with low probability, the chances of a disturbed analysis are that much greater. For the case $U(\mathbf{x}_0) = 1/500$, we can discard the observation and substitute $U(\mathbf{x}_0) = 10^{-4}$ because we have *a priori* knowledge that 10^{-4} is a better estimate of $U(\mathbf{x}_0)$ than 1/500. This is an example of a special case where it is possible to improve the estimates of X(x) by discarding outlying data points that do not conform to our *a priori* expectation.

A better approach to small amounts of data is to group the data according to sensible categories. For example, suppose we wish to examine the hypothesis that the responses tend to contain runs of correct items. We can group the vectors according to the longest run they contain, and sum the estimated probabilities $X(\mathbf{x})$ within each run length. This gives rare observations a chance to average out. One can compute predicted $X'(\mathbf{x})$ from $X^{(i)}$ (the probability the subject knows item i) under the independence hypothesis and use a chi-square test to evaluate the hypothesis.

Grouping observations is a means of matching the precision of the question asked to the precision of the data, and it is the most justifiable treatment of small numbers of observations.

Array Storage Sizes

The results obtained in the second part of this paper are more general than the results obtained in the first part. In fact, the earlier results can readily be derived from these more general equations

$$\mathrm{X}(\mathrm{x}) = \sum_{\mathrm{e}=\mathrm{x}} X(\mathbf{e}),$$

$$\mathrm{Y}(\mathrm{y}) = \sum_{\mathrm{e}=\mathrm{y}} Y(\mathbf{e}).$$

On the other hand, the computations involving serial positions are more complicated and time-consuming than those that do not and it is advisable to avoid them if they are not needed. Table I illustrates the size of the various arrays involved in the computations. A practical computational limit occurs when item and position scores are considered jointly and when the stimulus length exceeds eight or nine. For example, the number of possible entries for $W(\mathbf{u}, \mathbf{v})$ is 59,049 when the number of stimulus items is ten. If one does not use the fact that $\mathbf{u} \gtrsim \mathbf{v}$, but only that $\mathrm{u} \geqslant \mathrm{v}$, then for $m = 10$, one would have to reserve $(2^m)(2^{m-1} + 1) = 525, 312$ cells. And if one does

TABLE I

Array Sizes for the Various Computations[a]

	Summed scores (scalars)		Individual serial positions (vectors)	
Stimulus length	u	W(u, v)	$\mathbf{u}$	$W(\mathbf{u}, \mathbf{v})$
m	$m+1$	$(m+1)[(m/2)+1]$	2^m	3^m
2	3	6	4	9
4	5	15	16	81
6	7	28	64	729
8	9	45	256	6,561
10	11	66	1,024	59,049
12	13	91	4,096	531,441

[a] The number of response items m equals the number of stimulus items; the number of different item (or position) scores is shown under u; the number of possible different joint occurrences of u and v is shown under W. The corresponding numbers for vectors are shown under **u** and *W*.

not use the iterative procedure, the solution for $Z(\mathbf{x}, \mathbf{y})$ in terms of $W(\mathbf{u}, \mathbf{v})$ requires inversion of a $3^{10} \times 3^{10}$ matrix. In our experience, the full vector analysis can be used only for stimuli whose length does not exceed seven or eight.

The equations in this paper have been checked and verified on test data generated by Monte Carlo methods. Fortran programs for the various computations are described in an unpublished paper by Melvin J. Melchner.[4]

Reference

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[4] "Fortran Programs for Estimating Item and Order Information" is available from the author and has been deposited as Document No. NAPS 02801 with the National Auxiliary Publications Service, c/o Microfiche Publications, 440 Park Avenue South, New York, N.Y. 10016. A copy may be secured by citing the document number and remitting $8.75 for photocopy or $3.00 for microfiche. Advance payment is required. Programs in card form (and the paper) are available at cost from the Computing Information Service Group, Bell Labs, Murray Hill, New Jersey, 07974.