CHAPTER SIX

INTRODUCTION TO DYNAMICS

We begin to develop "participator dynamical systems" on environments supported by reflexive frameworks. We introduce the notions of action kernel and participator. For the cases of one and two participator systems, we give a description of the participator dynamics in the language of Markov chains. This chapter is motivational; it deals intuitively with very restricted cases. In the next chapter we consider a more general case.

1. Mathematical notation and terminology

The dynamics developed in this chapter makes use of several mathematical concepts from the theory of Markov chains. In this section we collect basic terminology and notation for the convenience of the reader.[1] We assume a familiarity with the notions of conditional probability and expectation.

Let $(E, \mathcal{E})$ be a measurable space. The set of measurable functions $f\colon E \to \mathbf{R}$ that are bounded is denoted by $b\mathcal{E}$, and the set of measurable nonnegative functions by $\mathcal{E}_+$.

Recall from chapter two that a kernel P on E is said to be ***positive*** if its range is in $[0, \infty]$. It is called a ***transition probability*** or a ***submarkovian kernel*** if $P(e, E) \leq 1$ for all $e \in E$. It is called ***markovian*** if $P(e, E) = 1$ for all $e \in E$. The abbreviation T.P. is sometimes used for transition probability. If P is a positive kernel and $f \in \mathcal{E}_+$, for example, then P can be viewed as an ***operator*** taking f to the function Pf defined by $Pf(e) = \int_E P(e, \mathrm{d}h)f(h)$.

[1] For more background, beginning readers might refer to Breiman (1969) or Narayan Bhat (1984). For advanced readers we suggest Revuz (1984).

Similarly, if ν is a positive measure on $\mathcal{E}$, then P can be viewed as the operator on measures $\nu P(A) = \int_E \nu(\mathrm{d}h)P(h,\, A)$ for $A \in \mathcal{E}$. The ***composition*** or ***product*** of two positive kernels P and Q is the kernel $PQ(e,\, A) = \int_E P(e,\, \mathrm{d}h)Q(h,\, A)$. The n-fold product of a kernel P with itself is denoted P_n.

Let $(\Omega, \mathcal{F}, P_0)$ be a probability space and $Z = \{Z_n\}_{n\geq 0}$ a sequence of random variables $Z_n\colon \Omega \to E$. Such a structure is called a ***stochastic process*** with ***base space*** $(\Omega, \mathcal{F}, P_0)$ and ***state space*** E. A sequence $\{\mathcal{G}_n\}_{n\geq 0}$ of subσ-algebras of $\mathcal{F}$, such that $\mathcal{G}_n \subset \mathcal{G}_{n+1}\ \forall n$, is called a ***filtration*** on $(\Omega, \mathcal{F})$. Let $\mathcal{F}_n = \sigma(Z_m,\, m \leq n)$ and $\mathcal{G}_n$ be a filtration such that $\mathcal{G}_n \supset \mathcal{F}_n$ for every n. The sequence $Z = \{Z_n\}_{n\geq 0}$ is called a ***Markov chain with respect to the filtration*** $\{\mathcal{G}_n\}_{n\geq 0}$ if, for every n, the σ-algebras $\mathcal{G}_n$ and $\sigma(Z_m,\, m \geq n)$ are conditionally independent with respect to Z_n; i.e., if for every $A \in \mathcal{G}_n$ and $B \in \sigma(Z_m,\, m \geq n)$, $P_0[A \cap B|\, Z_n] = P_0[A|\, Z_n]P_0[B|\, Z_n]$ a.s. The σ-algebras $\mathcal{F}_n$ are referred to as "past" σ-algebras. When we say simply that Z ***is a Markov chain (with base space*** $(\Omega, \mathcal{F}, P_0)$***)*** we mean that it is so with respect to the past algebras $\mathcal{F}_n$. Intuitively, a sequence of random variables is a Markov chain if the probabilities for passing into the next state are completely determined by the current state of the system.

A sequence $Z = \{Z_n\}_{n\geq 0}$ of random variables is called a ***homogeneous Markov chain with respect to the filtration*** $\{\mathcal{G}_n\}$ ***with transition probability*** P if, for any integers m, n with $m < n$ and any function $f \in \mathrm{b}\mathcal{E}$, we have $E_0[f(Z_n)|\, \mathcal{G}_m] = P_{n-m}f(Z_m)$, P_0 a.s., where E_0 denotes the mathematical expectation operator with respect to P_0. The probability measure ν defined by $\nu(A) = P_0[Z_0^{-1}(A)] \equiv P_0[Z_0 \in A]$, for $A \in \mathcal{E}$, is called the ***starting measure.***

Let P be a T.P. on E. It is customary to extend the state space $(E,\, \mathcal{E})$ to the space $(E_\Delta, \mathcal{E}_\Delta)$, where Δ is a point not in E called the ***cemetery,*** $E_\Delta = E \cup \{\Delta\}$, and $\mathcal{E}_\Delta = \sigma(\mathcal{E},\, \{\Delta\})$. P extends to a markovian kernel on $(E_\Delta,\, \mathcal{E}_\Delta)$ by setting $P(e,\, \{\Delta\}) = 1 - P(e,\, E)$ if $e \neq \Delta$, and $P(\Delta,\, \{\Delta\}) = 1$. A ***canonical probability space*** is the space $(\Omega, \mathcal{F}, P_0)$ where $\Omega = \prod_{n=0}^\infty E_\Delta^{(n)}$, and $E_\Delta^{(n)}$ is a copy of E_Δ; where the σ-algebra $\mathcal{F}$ is generated by the semi-algebra of ***measurable cylinders*** of Ω (namely sets of the form $\prod_{n=0}^\infty A_n$, where $A_n \in \mathcal{E}_\Delta^{(n)}$, and A_n differs from $E_\Delta^{(n)}$ for only finitely many n); and where P_0 is a ***probability*** measure. A point $\omega = \{\omega_n,\, n \geq 0\}$ of Ω is called a ***trajectory*** or ***path.*** The mapping $Z_n\colon \Omega \to E_\Delta^{(n)}$ taking $\omega = (\omega_0, \omega_1, \omega_2, \ldots) \in \Omega$ to its nth entry ω_n is called the n***th coordinate mapping.*** If the sequence $Z = \{Z_n\}$ of coordinate mappings on the canonical probability space forms a homogeneous Markov chain with T.P. P, we call it the ***canonical Markov chain with T.P.*** P.

The ***shift operator*** θ is the point transformation on Ω defined by $\theta(\omega_0,\, \omega_1,$

$\ldots, \omega_n, \ldots) = (\omega_1, \omega_2, \ldots, \omega_{n+1}, \ldots)$. We write θ_n for the n-fold iteration of θ: $\theta(\omega_0, \omega_1, \ldots) = (\omega_n, \omega_{n+1}, \ldots)$. A ***stopping time*** T of the canonical Markov chain Z is a random variable defined on $(\Omega, \mathcal{F})$ with range in $\mathbf{N} \cup \{\infty\}$ and such that for every integer n the event $\{T = n\}$ is in $\mathcal{F}_n$. ($\mathbf{N}$ is the set of natural numbers including 0.) The ***σ-algebra associated with*** T is the family $\mathcal{F}_T$ of events $A \in \mathcal{F}$ such that for every n, $\{T = n\} \cap A \in \mathcal{F}_n$. Notice that then the random variable $Z_{T(\omega)}(\omega)$ is $\mathcal{F}_T$-measurable.

Let G be a group that is locally compact with countable basis (LCCB), and let $\mathcal{G}$ denote the σ-algebra of its Borel sets. Given probability measures μ_1, μ_2 on $\mathcal{G}$, their ***convolution*** $\mu_1 * \mu_2$ is defined to be the probability measure which assigns to $K \in \mathcal{G}$ the measure $(\mu_1 * \mu_2)(K) = \int\int 1_K(x+y)\mu_1(dx)\mu_2(dy)$. A ***right (left) random walk on*** G is a Markov chain with state space $(G, \mathcal{G})$ and transition probability $\epsilon_g * \mu$ $(\mu * \epsilon_g)$, where μ is a probability measure on $(G, \mathcal{G})$ which is called the ***law*** of the random walk, and ϵ_g is Dirac measure supported at the point $g \in G$. On an abelian group there is only one random walk of law μ, and it is invariant under translations.

2. Fundamentals of dynamics

The conclusion of an observer O's perceptual inference is represented, as we have discussed, by a probability measure $\eta(s, \cdot)$. This conclusion is true in a given semantics, according to Definition **4**–3.5, if $\eta(s, \cdot)$ is the actual regular conditional distribution, given s, of the measurable functions X_t (defined in **4**–3.1). $\{X_t\}$ is a sequence of random variables indexed by a discrete time t, taking values in configuration space X, and whose domain is some unspecified probability space Ω. In extended semantics (**4**–4) there is a set $\mathcal{B}$ of objects of perception; for each t, a value of X_t is associated with an interaction of O with an element of $\mathcal{B}$. These interactions are called channelings. In the case of an environment supported by a reflexive framework (**5**–2.6) we have a set of observers $\mathcal{B}$ which is also the set of objects of perception for each of its members. At each instant of "reference" time (which, as we shall see, is not the time t of the random variables X_t) the totality of channeling interactions at that instant is described by a subset L of $\mathcal{B}$ and a relation $\tilde{\chi}$ on L as in **5**–3.

We now begin to construct a class of models for environments supported by reflexive frameworks; these models are called "participator dynamical systems." We do this using entities called "participators"; a participator manifests as an observer in $\mathcal{B}_E$ at each instant of reference time. The subset D of $\mathcal{B}$ at reference time n always contains the set of participator manifestations at time n. The determination of D and χ can be discussed in terms of participators (we discuss this in **7**–2). In the process of this development, an analytical viewpoint emerges in which the participators themselves are the center of attention.

This chapter is informal; for clarity we present many of the ideas in special cases. In the next chapter we provide a formal development.

The motivation for dynamics

Consider two observers, A and B, in a reflexive framework $(X,\ Y,\ E,\ S,\ \pi_\bullet)$ of the type shown in Figure **5**–2.9. Recall (from **5**–2.8) that this means there exist points $a, b \in E$ such that $A = (X,\ Y,\ E,\ S,\ \pi_a,\ \eta_A)$ and $B = (X,\ Y,\ E,\ S,\ \pi_b,\ \eta_B)$, where $\eta_A,\ \eta_B$ are some conclusion kernels. We depict this in Figure 2.1, where the observers A and B channel to each other. Each makes an inference about the perspective map of the other, i.e., about the point of E that represents the perspective of the other. Figure 2.1 shows the premise $s = \pi_a(b)$ of A's perceptual inference, and the ray of configuration points x such that $\pi_a(x) = s$ (labelled in the figure as $\pi_a^{-1}\{s\}$). A's conclusion measure η_A is supported on the set $\xi = \pi_a^{-1}\{s\} \cap E$, which includes b. But ξ includes infinitely many perspectives other than B's as well, some of which are indicated by the smaller dashed circles with numbers above. Thus A is faced with perceptual uncertainty: What was the perspective of the observer that channeled? Was it 1, 2, b, 3, …? In general, A cannot pick just one perspective as the answer to this question. Instead A concludes that it is perspective 1 with probability P_1, perspective 2 with probability P_2, perspective b with probability P_b, and so on. This is the content of A's conclusion measure $\eta_A(s, \cdot)$.

How is A's conclusion measure $\eta_A(s, \cdot)$ to be chosen? On what basis can A conclude that the other observer's perspective was 1 with probability P_1, 2 with probability P_2, etc? The answer we give is roughly as follows. A markovian dynamics of perspectives naturally arises in the context of reflexive frameworks. That observers in the framework perceive truly means that their η's should be related to the asymptotic behavior of this dynamics. Intuitively, the probability assigned by $\eta(s, \cdot)$ to a point $e \in E$ should be a conditional probability derived from the frequency with which the perspective corresponding to that e is adopted by participators in the given dynamical context. In this sense, the given dynamics plays the role of the "environment" in which these observers are embedded. To make these ideas more precise, we begin by

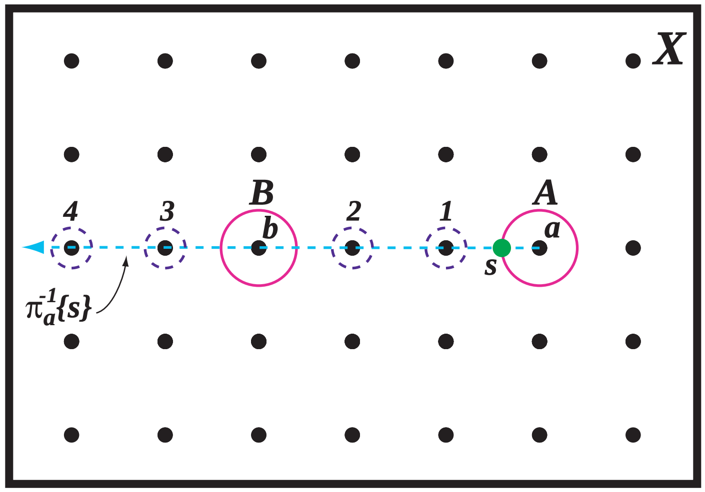

FIGURE 2.1. *Perceptual uncertainty in a reflexive framework.*

discussing how a dynamics of perspectives arises on reflexive frameworks.

When A and B channel, the premise s of A's perceptual inference greatly restricts what A can conclude about B's perspective. Yet A has, in general, infinitely many choices remaining, for B's perspective could be any in $\pi_a^{-1}\{s\} \cap E$. Suppose A and B retain their perspectives after channeling. Then if they channel again A has precisely the same set of choices—and the same ambiguity—regarding B's perspective as before. In other words, if the observers do not alter their perspectives after an observation then there is no point to further observations. A can channel with B as many times as you like, but the same premise s will result every time, and with it the same ambiguity of interpretation. Moreover, should A and B never change perspectives the whole question of how η_A is chosen would be trivial: the ideal $\eta_A(s, \cdot)$ would be Dirac measure supported on b, and $\eta_A(s', \cdot)$ for $s' \neq s$ would need not be defined. Indeed, the construction of reflexive frameworks would be pointless.

Let us, then, allow observers in a reflexive framework to change perspective following a channeling. That is, let us allow some kind of dynamics of perspectives on reflexive frameworks. Several questions immediately arise. How shall observers change perspective? Since the perspective of an observer in a reflexive framework (together with its conclusion measure) is its only means of individuation, does not a change in perspective actually mean a change in observer? If so, then what is it that is manifesting itself as a different observer at each step of the dynamics? Furthermore, dynamics requires sequence. What is the formal structure of this sequence? What is the formal structure of the dynamics? How, precisely, shall η be related to the resulting dynamics? We consider these questions in turn here, and in succeeding chapters.

Action kernels

How should we allow observers to change perspective in reflexive frameworks? There are two basic issues. First, what information should be used to select the new perspective of an observer after a channeling? Second, should the new perspective be chosen deterministically or probabilistically? We discuss these issues in the context of Figure 2.2. This figure shows observers A and B channeling to each other. In consequence of the channeling, A's premise is s_A, and B's premise is s_B. The figure shows A changing its perspective from π_a to $\pi_{a'}$, and B changing from π_b to $\pi_{b'}$. Of course, after these changes A and B are no longer the same observers since they no longer have the same perspectives. We denote the new observers A' and B'. As can be seen in the figure, the only information available to choose A's new perspective is its current perspective and the premise s_A. Similarly, mutatis mutandis, for B. Therefore, for maximum generality, we assume that an observer's next perspective is some function of its current perspective and current premise. Shall this function be deterministic or probabilistic? Again, for maximum generality, we assume that the next perspective is chosen probabilistically, according to some measure. (The deterministic case is the special case that the measure governing the choice of next perspective is a Dirac measure.) Further, we assume this measure to be a probability measure; after we introduce the notion of ***participator,*** we will interpret this assumption.

In light of these considerations, we could propose that the change in perspective of an observer A should be governed by a probability measure that is selected based on A's current perspective and current premise. However, a sense of symmetry suggests that A's probability measure should depend not on its absolute perspective, but on the "difference" between its perspective and that of B. Symmetry also suggests that the probability that A moves to A' should depend only on the "difference" in perspective between A and A'. Now to talk about differences of perspective in E requires some structure on E. For instance, E might be a principal homogeneous space for some group of "translations." More generally, the minimum structure necessary here is a

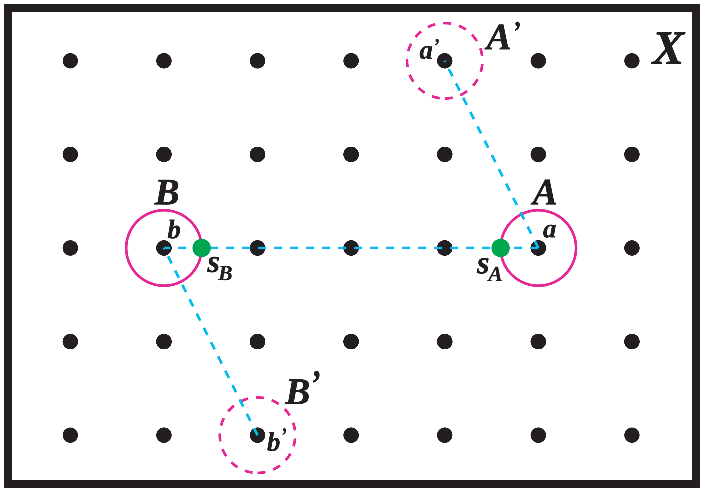

FIGURE 2.2. *Changing perspective on a reflexive framework.*

symmetric framework (Definition **5**–5.1). However, since the purpose of this chapter is to introduce basic ideas of observer dynamics, we defer (until chapter seven) a systematic presentation at this level of generality. Throughout this chapter we assume, for simplicity,

Assumption 2.3. We are working in a symmetric framework (X, Y, E, S, G, J, π) in which $G = X$ is an abelian group written additively and $J = E$ is a subgroup. Equivalently, we can say that $(X, Y, E, S, \pi_\bullet)$ is a reflexive framework in which X is an abelian group, $E \subset X$ is a subgroup, and there exists a map $\pi\colon X \to Y$ such that for each $e \in E$, $\pi_e(x) = \pi(x - e)$.

Thus we can speak of "differences in perspective" without thinking twice. The reader may rely for intuition on examples like Example **5**–4.3: one can think of X as $\mathbf{R}^n$ (with vector addition as the group operation) and E as a measure zero subgroup thereof.

We return now to the question of the probability measure governing changes in perspective. In light of our assumptions, this is a measure on a group E, telling how probable are various translations from the current perspective. We can capture the dependence of the measure on the observer's current premise by associating to each premise s of the observer a measure on the group of translations that acts on E. The appropriate mathematical device to do this is a kernel Q that we call an *action kernel.* For each premise s, the measure $Q(s, \cdot)$ is a probability measure on the group of translations that acts on E, (the group being, in this chapter, E itself).

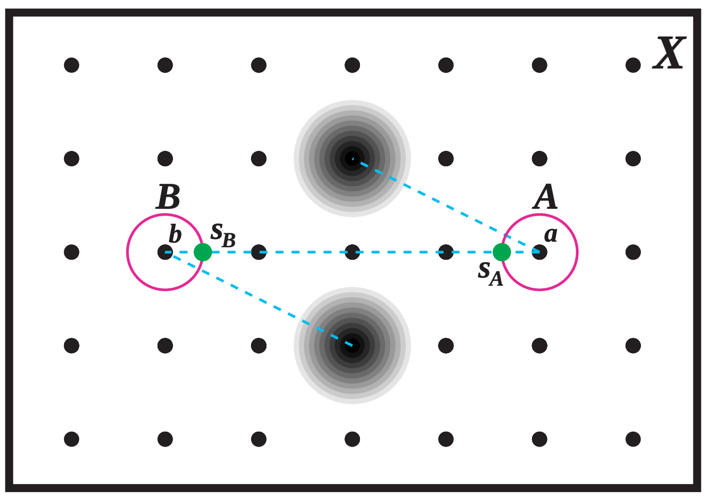

FIGURE 2.4. *Action kernels. The shading of the upper circular region represents the density of the probability distribution* $Q_A(s_A, \cdot)$*. Similarly, the shading of the lower circular region represents the density of the probability distribution* $Q_B(s_B, \cdot)$.

These notions are illustrated in Figure 2.4. Once again, two observers A and B channel with each other. A's premise is s_A. The measure $Q_A(s_A, \cdot)$, derived from A's action kernel Q_A, is depicted by a shaded disk with a dashed line drawn from A to the center of the disk. The darkness of a region within this disk encodes the probability that A will adopt a perspective in that region as its next perspective. Darker regions are more probable than lighter ones. A's expected new perspective happens, in the case illustrated, to be the perspective represented by the center of the disk. In general there will be some probability that an observer does not change its perspective after a channeling. (However for pictorial clarity the disk is not drawn large enough to include A's own perspective.)

Definition 2.5. Under Assumption 2.3, an ***action kernel*** is a markovian kernel $Q: E \times \mathcal{E} \to [0, 1]$ such that $Q(e, \cdot) = Q(e', \cdot)$ if $\pi(e) = \pi(e')$. Given Q, to each $e_1 \in E$ we can associate a kernel $Q_{e_1}: E \times \mathcal{E} \to [0, 1]$ by $Q_{e_1}(e, \Gamma) = Q(e - e_1, \Gamma)$.

Suppose Q is an action kernel. Since $Q(e, \cdot)$ depends only on $\pi(e)$ we could equally well define it as a kernel $S \times \mathcal{E} \to [0, 1]$. In fact we will sometimes write $Q(s, \cdot)$; this will mean $Q(e, \cdot)$ for any e such that $\pi(e) = s$. Similarly $Q_{e_1}(e, \cdot)$ depends only on $\pi_{e_1}(e)$. The interpretation of the action kernel is as follows: $Q(e, \Gamma)$ is the probability that the observer will change perspective by an increment in the set Γ, given that it channeled with an observer whose perspective differed from its perspective by e. If the first observer is at e_1, then $Q_{e_1}(e_1 + e, \Gamma)$ is an equivalent way to write this. The terminology "action kernel," when used for a given kernel $Q: E \times \mathcal{E} \to [0, 1]$, signals our intention to consider the family of kernels $\{Q_e\}_{e \in E}$.

Participators

In our discussion of action kernels we have spoken as though an observer in a reflexive framework could change its perspective map π. We said, for instance, that an action kernel gives the probabilities with which an observer might adopt various new perspectives. Now this way of speaking, though convenient, cannot be correct; the definition of observer does not permit a given observer to change its perspective. On the contrary, the definition requires an observer to have a fixed perspective map $\pi: X \to Y$. Therefore, the formal entity that changes perspective according to the dictates of an action kernel is not itself an observer. Instead this entity manifests itself at each instant as an observer in the context of a reflexive framework. This new formal entity we call a "participator."

Definition 2.6. A ***participator*** on a reflexive framework $(X, Y, E, S, \pi_\bullet)$ (under Assumption 2.3) is a triple, $(\xi, \{Q(n)\}_n, \{\eta(n)\}_n)$, where n varies over the nonnegative integers, ξ is a probability measure on E, each $Q(n)$ is an action kernel, and each $\eta(n)$ is a family of interpretation kernels for the reflexive framework. (That is, $\eta(n) = \{\eta_e(n)\}_{e \in E}$, where, for each $e \in E$, $(X, Y, E, S, \pi_e, \eta_e(n))$ is an observer.) If all the $Q(n)$ are equal to a fixed action kernel Q, we denote the participator simply by $(\xi, Q, \eta(n))$, and call it a ***kinematical participator with action kernel*** Q. If, for some n, a participator $A = (\xi, \{Q(n)\}_n,$

$\{\eta(n)\}_n$) on a reflexive framework $(X, Y, E, S, \pi_\bullet)$ has perspective π_n, then we call the observer $A_n = (X, Y, E, S, \pi_n, \eta(n))$ the ***manifestation of** A **at time** n***. We also say that A ***manifests*** as A_n. A ***preparticipator*** is a pair $(\xi, \{Q(n)\}_n)$ with ξ and $Q(n)$ as in a participator.

The formal definition of participator is based upon the following intuitions. A participator must have a first perspective; this is the purpose of ξ. The probability measure ξ on E, called the ***initial measure*** of the participator, governs the choice of the first perspective of the participator. When we say that a participator is initially "at e" or "has perspective e" we mean a participator for which ξ is Dirac measure at e; formally, we write $\xi = \epsilon_e$. A participator must also have a means of changing perspective; this is the purpose of the action kernels $Q(n)$. The changes in perspective are discrete and sequential, with respect to a notion of time that we discuss shortly. The notation means that the nth change of perspective in this sequence is governed by the action kernel $Q(n)$. Since the action kernels give probabilities for change of perspective conditioned by premises arising from channelings, the perspective changes of participators are probabilistic and are driven by channelings. The terminology "kinematical participator," for the special case when all the $Q(n)$ are identical, indicates that this case gives rise to systems with a property analogous to constant velocity. This does not mean that the motion of the participators is "linear" in the usual geometric sense of the word. Rather, it means that the instantaneous state-change data (in this case, given by the action kernel) is time invariant.

We discuss shortly a dynamics of perspectives that arises from the mutual observations of an ensemble of participators in a common reflexive framework. This dynamics is a Markov chain whose state space is a product of copies of E, one for each participator in the ensemble. In this chapter we consider a simplified version of the dynamics which is determined entirely by the action kernels and initial measures of the participators. To specify a (canonical) Markov chain on some space one need only give its initial measure and transition probability. The initial measure of the markovian dynamics of perspectives is simply the product of the initial measures of the participators; we study the transition probability in chapter seven. In the special case of kinematical participators the resulting Markov chains are ***homogeneous.*** In this case we will sometimes use the word ***kinematics*** rather than ***dynamics.***

A participator dynamics on a reflexive framework incorporates a nondualistic model of extended semantics. There is some set $\mathcal{B}$ of observers in the framework; $\mathcal{B}$ serves as the objects of perception for each observer in the framework. In participator dynamics the set $\mathcal{B}$ has a special property; this property consists in a precise condition on the subset $\mathcal{B}_E$. Let us begin with a fixed set K of participators; say it is this set of participators whose dynamical interaction constitutes the given participator dynamical system. *$\mathcal{B}_E$ is then the set of all possible instantaneous manifestations for the participators in K.* To be precise, suppose each $j \in K$ is represented $(\xi_j,\ \{Q_j(n)\}_n,\ \{\eta_j(n)\}_n)$. Then

$$\mathcal{B}_E = \bigcup_n \{(X, Y, E, S, \pi_e, \eta_j(n)) \,|\, e \in E, j \in K\}. \tag{2.7}$$

Taking a union over the various instants of time n implies that we do not distinguish the observers $(X,\ Y,\ E,\ S,\ \pi_e,\ \eta_j(n))$ and $(X, Y, E, S, \pi_e, \eta_j(n'))$ (for e and j fixed) if it happens that the kernels $\eta_j(n)$ and $\eta_j(n')$ are equal for distinct times n, n'. By contrast, for $j \neq j' \in K$ and for a given $e \in E$, the observers $(X,\ Y,\ E,\ S,\ \pi_e,\ \eta_j(n))$ and $(X,\ Y,\ E,\ S,\ \pi_e,\ \eta_{j'}(n))$ are counted as distinct elements of $\mathcal{B}$, even if the kernels $\eta_j(n)$ and $\eta_{j'}(n)$ are the same.

Definitions of $\mathcal{B}_E$ other than 2.7 are possible. For example we could have taken a *disjoint* union over n rather than ordinary union as we did in 2.7. This means that the manifestations of a given participator at distinct moments n, n' would always be viewed as distinct elements of $\mathcal{B}$, even if the perspectives and conclusion kernels of the two observers were identical. This would allow the present manifestation of a participator to interact with a previous manifestation of the same participator—let us call this a "memory interaction"—in a manner which permits keeping track of the distinct times. However, using 2.7 it is still possible for a present and a past manifestation of a single participator to interact. The difference is that now, if these two manifestations happen to be identical as observers, then they are also considered identical as objects of perception; they are represented by the same element $b \in \mathcal{B}$. Thus the interaction in question is characterized by b channeling with itself. (According to **5**–3.2 and **5**–3.4 this means that at the given instant there is a distinguished subset $D \subset \mathcal{B}$ containing b, and an involution χ of D such that $\chi(b) = b$.) In other words, in the context of 2.7, a memory interaction may be analytically indistinguishable from a self-channeling, whereas in the alternate (disjoint union) approach memory interaction and self-channeling are always distinct. Whether this difference is theoretically significant is an open question.

We can now interpret the requirement that action kernels are markovian (2.5), i.e., that if Q is an action kernel then for each $e \in E$ the measure $Q(e, \cdot)$ is a probability measure on E. This means that the set of participators which manifest themselves as observers is the same set at each instant of time:

participators do not appear or disappear while a scenario is running. To see this, recall that if Q is the action kernel of a participator A, the measure $Q(e, \cdot)$ assigns probabilities to A's perspective at time $n+1$ (given that, at time n, A channeled with some participator whose perspective is e). And if Q is not markovian, i.e., if $Q(e, E) < 1$, then there is positive probability that A has no perspective at time $n+1$, so that it is not manifested as an observer in the framework at that time. However, though A must manifest as an observer at each time n, A's manifestation need not channel at each time n. In other words, the subset L of $\mathcal{B}$ which is the domain of the channeling relation at time n may be a proper subset of the set of all participator manifestations at time n. We see, then, that the markovian requirement on action kernels is a matter of convention, not a restrictive assumption: since we do not require the participators to channel at every instant, and since the dynamics is driven by channeling, the net effect on the dynamics is the same whether a participator does not manifest at time n, or manifests but does not channel at time n.

Reference and proper times

Dynamics requires some notion of time or sequence. Our notion of time in the context of participator dynamics is guided in part by the ideas of Einstein:

> " The experiences of an individual appear to us arranged in a series of events; in this series the single events which we remember appear to be ordered according to the criterion of 'earlier' and 'later,' which cannot be analyzed further. There exists, therefore, for the individual, an I-time, or subjective time. This in itself is not measurable. I can, indeed, associate numbers with the events, in such a way that a greater number is associated with the later event than with an earlier one; but the nature of this association may be quite arbitrary."[2]

The only events in a reflexive framework with which to associate numbers are the discrete acts of observation and the consequent changes in perspective. To each participator, then, we assign a number, called the "proper time" of that participator, such that the number increases only when the participator makes an observation. Every channeling that involves that participator increases its proper time. Thus discrete acts of observation constitute the units of subjective time in this framework. We will give a more formal treatment

[2] Einstein (1956), p. 1.

of proper time in chapter seven; the examples we present in this chapter are simplified (artificially) so that the proper time of each participator coincides with "reference time" (defined below).

The setting of the dynamics described here is different from the space-time setting assumed in physics. In place of physical space we have the space of possible observer perspectives, and in place of physical time we have the sequence of discrete observations of participators.

A particular channeling may not include the perspectives of some participators in the dynamics. In this case the proper times of the excluded participators are not increased, but the proper times of the others are increased. Therefore, even if the proper times of all participators begin with the same value, say zero, their proper times will eventually differ due to channelings that exclude some participators. We cannot then, in general, take the proper time of any particular participator to be the time parameter of the markovian dynamics of the ensemble. For this we need a time parameter that increases for every channeling whether or not that channeling includes a particular participator. This time parameter we call "reference time." In a given dynamical setting in which we have a fixed set K of participators, we may take the reference time to be a copy of the nonnegative integers, called "R," which is the domain of the time index "n" of Definition 2.6 for all the participators in K. Thus in speaking of reference time we are making the assumption that these indices have a common domain.

The reference time in a given dynamical context (corresponding to a set K of participators) is not the same as the *active time* in the sense of **4**–2.1 for the observers in the set $\mathcal{B}_E$ of 2.7. In fact, reference time is associated to a set of participators, not to a set of observers. And the reference time need not include those instants when the participators channel only to non-distinguished objects of perception. It need only include those instants when participator observations occur, and by the term 'observation' we always mean a channeling which results in a distinguished premise (which causes the output of a conclusion, etc.). Now a channeling with a non-distinguished object of perception may result in a distinguished premise ("false targets"), and an instant of time in which this occurs (for the manifestation of a participator) would have to be included in reference time. But if no distinguished premises occurred at the given instant for any of the participators, then that instant would be excluded from reference time.

Recall, by contrast, that since the active time of an observer indexes the X_t's, it consists by definition precisely of those instants when the observer receives *any* channeling, from a distinguished object of perception or not.

Definition 2.8. With the terminology of **5**–3, (i) a ***participator channeling sequence*** is a function, ζ, from the natural numbers to the space of channelings, $\zeta\colon \mathbf{N} \to \mathcal{I}$, with the following property. Let $\zeta(n) = (L_n,\, \tilde{\chi}_n)$, where $L_n \subset \mathcal{B}$ and $\tilde{\chi}_n$ is an involution of L_n. Let $K_n = (D_n, \chi_n)$ denote the distinguished part (**5**–3.4) of $\zeta(n)$. ***Then for each*** $n \in \mathbf{N}$, D_n ***is not empty.*** **N** is called the ***reference time*** for the sequence.[3] (ii) As in 2.6, let $A_n \in \mathcal{B}_E$ denote the manifestation of participator A at reference time n. To each participator A in the dynamics is associated its ***proper time,*** $\mathcal{T}_A\colon \mathbf{N} \to \mathbf{N}$, defined inductively as follows:

$$\mathcal{T}_A(n) = \begin{cases} 0 & \text{if } n = 0 \\ \mathcal{T}_A(n-1) + 1 & \text{if } A_n \in D_n \text{ and } \pi_{\Phi(A_n)}(\Phi(\chi_n(A_n))) \in S \\ \mathcal{T}_A(n-1) & \text{otherwise.} \end{cases}$$

At every instant of reference time, the proper time of at least one participator is increased. Definition 2.8 says that the unit of subjective time for a participator is a single act of channeling, i.e., the performance of a single perceptual inference. Since at any step of reference time some participator manifestations may not channel, it follows that the proper times for different participators vary: proper time is relative to the participator. In fact it will be seen in chapter eight that, given any ensemble of participators, each participator's proper time is a ***stopping time*** for the associated dynamical Markov chain.

According to Definition 2.8, the proper time of a participator A increases not only if its manifestation channels with a distinguished object of perception, but also if it channels with a false object. A false object is an object of perception $B_n \in \mathcal{B} - \mathcal{B}_E$ such that $\pi_{\Phi(A_n)}(\Phi(B_n)) \in S$. If B_n is a false object then, using the terminology of **2**–3, $\Phi(B_n)$ is a false target. Channelings with false objects affect participator dynamics since participators, unable to distinguish false objects from true, change perspective according to their action kernels upon channeling with false objects. In this book we attempt no serious investigation of the role of such channelings in participator dynamics. In fact we ignore false objects and assume that, at each instant of reference

[3] Thus a participator channeling sequence assigns a nonempty channeling to each instant of reference time. At every instant of reference time the manifestation of at least one participator channels. In this book we consider only those sequences such that the sets D_n have some fixed maximum size.

time, participator manifestations channel only with other participator manifestations. (As an informal justification for this one might assume that the statistical properties of the action kernels somehow take into account these extraneous channelings.) This is the content of the following "closed system" assumption:

Assumption 2.9. *Closed system.* For each reference time n, D_n is contained in the set of participator manifestations at time n.

One further assumption should be noted. We conceive of the change of perspectives of participators on a reflexive framework as probabilistic. However, we have not given explicit details of the underlying probability spaces on which the dynamical mechanism depends. Our proposal for the underlying framework will be made in the next chapter. Here we note only the following characteristic:

Assumption 2.10. *Independent action.* At any instant of reference time, and given the current perspectives of all participators and the current channeling involution, the perspectives of the participators at the next instant of reference time are independent random variables.

For example, suppose we have three participators A, B and C with action kernels Q_A, Q_B and Q_C respectively, and with channeling involution $\chi = \{(A, B)\}$ (so that C is not channeled to). Then the probability that, at the next instant, $A \in \Gamma_A$, $B \in \Gamma_B$, and $C \in \Gamma_C$ is

$$Q_{A,e_A}(e_B, \Gamma_A) Q_{B,e_B}(e_A, \Gamma_B) 1_{\Gamma_C}(e_C).$$

That is, we need simply take a product of the appropriate probabilities for the individual participators.

3. Kinematics of a single participator

In this section we consider the kinematics of perspectives that arises in a system consisting of a single kinematical participator. We find that this kinematics is a random walk. In the next section we consider a kinematics of two participators. We consider the general case in the next chapter.

Consider a single participator on a symmetric framework (X, Y, E, S, G, J, π) satisfying Assumption 2.3. Let $\xi = \epsilon_e$, $e \in E$. The first manifestation of this participator then has perspective map π_e, defined by $\pi_e(g) = \pi(g - e)$, $g \in X$. The only channeling possible, since there is but one participator, is a "self channeling," viz., a channeling in which $\chi(e) = e$. The participator's premise is then $\pi_e(e)$, i.e., $\pi(0)$, where 0 denotes the identity element of our additive abelian group E. This applies to each instant of the participator's proper time and, since there are no other participators, the system is inert at all other instants. It follows from this that the same perceptual premise $s_0 = \pi(0) \in S$ obtains at each step of the kinematics. And, denoting by Q the action kernel of the participator, this implies that the same probability measure $Q(s_0, \cdot)$ for the next perspective obtains at each step of the kinematics. This implies that the kinematics is a random walk of law $Q(s_0, \cdot)$ with respect to the discrete time which is the participator's proper time and, in this special case, the reference time.

4. Kinematics of pairs

We now consider a system involving two kinematical participators. In such a system each participator might channel with itself, with the other participator, or not at all, at each step of reference time. In this section we assume for simplicity that each participator channels with the other at each step of the kinematics. In the next chapter we consider the general case.

Again we are in the situation of Assumption 2.3. When two participators, A and B, observe each other, each changes its perspective according to its action kernel. This leads to a new ***difference*** in their perspectives. This change in the relationship between their perspectives is governed by a kernel P which we can define as follows: for each $e \in E$ and $\Gamma \in \mathcal{E}$, $P(e, \Gamma)$ is the probability that, as the result of a change in their perspectives, the new perspective of B relative to A (i.e., the difference of their new perspectives) will lie in the set Γ, given that the present difference in their perspectives is e. We can compute P from the action kernels of the individual participators as illustrated in Figure 4.1. The figure shows two participators with initial perspectives a and b. The perspective of B relative to A is e (that of A relative to B is $-e$). After observing, A changes perspective by an amount dk and B changes perspective by an amount dh. This leads to a new difference in perspective $e-dk+dh$ (or $-(e-dk+dh)$).

Let Q and R denote the action kernels of the participators whose current perspectives are a and b respectively. Then the probability that A changes perspective by an amount dk given that B's perspective differs from A's by an amount e is $Q(e,\, dk)$. Similarly, the probability that B changes perspective by an amount dh given that A's perspective differs from B's by an amount $-e$ is $R(-e,\, dh)$. The probability of the joint event that A changes by dk and B by dh is, by Assumption 2.10, $Q(e,\, dk)R(-e,\, dh)$. That is, the probability that the new difference in perspective is $e-dk+dh$, given that the old difference in perspective was e, is given by $Q(e,\, dk)R(-e,\, dh)$. Thus, to determine what is the probability that the new difference in perspective lies within a region $\Gamma \in \mathcal{E}$, we simply find the measure of the region $\{(k,\, h) \in E \times E \,|\, e-k+h \in \Gamma\}$ with respect to the product measure $Q(e,\, dk) \otimes R(-e,\, dh)$ on $E \times E$. This is the same as the integral

$$\int_{E\times E} 1_\Gamma(e-k+h)Q(e,\, dk)R(-e,\, dh);$$

we conclude that $P(e,\, \Gamma)$ is this integral.

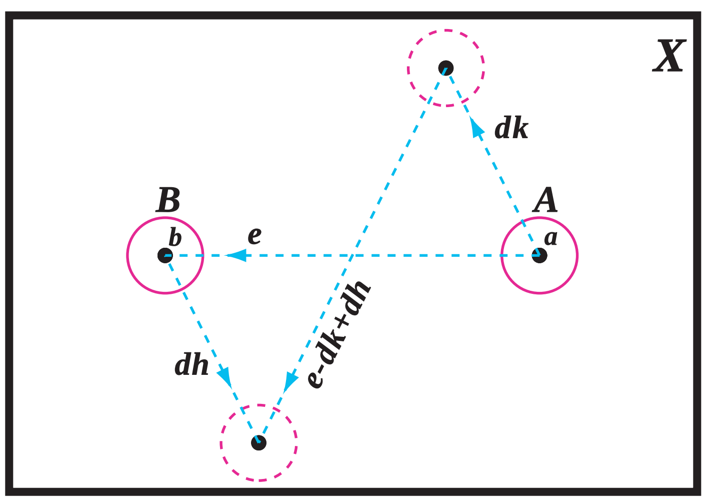

FIGURE 4.1. *Two participators change perspective.*

Note that P is time independent (assuming, as we do, that Q and R are) and is also independent of the absolute perspective. Thus we can summarize:

4.2. Suppose A and B are participators with action kernels Q and R respectively. Assume a channeling sequence where A channels only to B and vice versa. Then the proper times of A and B are the same. With respect to this proper time the successive perspectives of B relative to A (i.e., the successive differences in their perspectives) form a homogeneous Markov chain with state space E and transition probability P given by $P(e, \Gamma) = \int_{E\times E} 1_\Gamma(e-k+h)Q(e, dk)R(-e, dh)$.

The dependence of P on the action kernels Q and R can be conveniently and suggestively expressed in terms of a natural "bracket operation" which is derived from convolution of measures.

First, recall that if α, β are measures on the group $(E, \mathcal{E})$, then ***the convolution of*** α ***with*** β**,** denoted $\alpha * \beta$, is the measure on $(E, \mathcal{E})$ defined by

$$\alpha * \beta(\Gamma) = \int_{E\times E} 1_\Gamma(k+h)\alpha(dk)\beta(dh) \qquad (\Gamma \in \mathcal{E}).$$

Notation 4.3. If N is a kernel on $(E, \mathcal{E})$,

(i) $N^\dagger$ denotes the kernel $N^\dagger(e, \Gamma) = N(-e, -\Gamma)$, $(e \in E, \Gamma \in \mathcal{E})$;

(ii) $N_e(\cdot)$ denotes the measure $N(e, \cdot)$.

Definition 4.4. If Q and R are kernels on $(E, \mathcal{E})$, $[Q, R]$ is the kernel on $(E, \mathcal{E})$ given by

$$[Q, R](e, \Gamma) = (Q_e * R_e^\dagger)(e - \Gamma).$$

Proposition 4.5. With notation as above, $P = [Q, R]$.

Proof.

$$P(e, \Gamma) = \int_{E\times E} 1_\Gamma(e-k+h)Q(e, dk)R(-e, dh)$$

$$= \int_{E\times E} 1_{e-\Gamma}(k-h)Q(e,\, dk)R(-e,\, dh);$$

change variables so that h is replaced by $-h$:

$$= \int_{E\times E} 1_{e-\Gamma}(k+h)Q(e,\, dk)R(-e,\, -dh)$$
$$= \int_{E\times E} 1_{e-\Gamma}(k+h)Q(e,\, dk)R^\dagger(e,\, dh)$$
$$= (Q_e * R_e^\dagger)(e-\Gamma) = [Q,\, R](e,\, \Gamma).$$

For the moment, let P' denote the kernel for the Markov chain of perspectives of A relative to B. On the one hand, it is geometrically evident that $P'(e,\, \Gamma) = P(-e,\, -\Gamma)$ (where, as above, P denotes the kernel for the perspectives of B relative to A). On the other hand, from Proposition 4.5 we find that $P' = [Q,\, R]$. We conclude

Proposition 4.6. For any kernels Q, R,

$$[Q,\, R] = [R,\, Q]^\dagger.$$

This may also be verified directly from Notation 4.3 and Definition 4.4.

We close this section with several remarks. First, nothing prevents A and B from occupying the same perspective in E at a given instant. Second, the situation considered in this section, where each participator channels only to the other (and not to itself) is the opposite extreme of that treated in the previous section, where a participator channels only to itself. To make the comparison appropriate, imagine two kinematical participators A and B, each channeling only to itself. In this case we would get a Markov chain on $E \times E$; in each factor we would have a random walk, (one for A and one for B) as in the previous section. These random walks would be completely "uncoupled." In the situation treated in this section the perspectives of A and B are completely coupled: it is very unlikely that we would get anything resembling a random walk by looking at their sequences of states separately (or jointly). In the general setting, the question of the relative frequencies of cross-channelings and self-channelings in, say, a two participator dynamical system is described by an additional datum, called a τ-distribution, which we think of as describing the "informational conductivity" of E. Depending on the τ-distribution, the dynamical chain generated by an ensemble of participators with given action kernels will express some degree of coupling of the random walks each participator would undergo were there no cross-channelings. We will study this in more detail in the next chapter. The main idea here is that, given an ensemble of participators and a τ-distribution, a dynamical Markov chain is generated.

5. True perception among pairs

We have seen that a dynamics of perspectives arises naturally on reflexive frameworks. Intuitively, the purpose of the dynamics is to allow the participators to "perceive truly," i.e., to choose conclusion measures $\eta(s, \cdot)$ which in fact reflect the probabilities of events on the reflexive framework. Specifically, if participator A channels with B, leading to a premise s_A, then A should arrive at a conclusion measure $\eta_A(s_A, \cdot)$ which correctly describes with what probability the perspective of B, relative to A, lies in various subsets of $\pi_a^{-1}(s_A) \cap E$. In this section we specify conditions in which each participator, in a system of two mutually observing participators, can perceive truly the perspective of the other. Chapter eight addresses the issue of true perception formally and in greater generality.

We assume, as in the previous section, that there are no self-channelings; all channelings are cross-channelings. In chapter seven we consider more general dynamics, but several ideas are revealed by considering the simpler case.

We found in the last section that the kinematics of relative perspectives for two participators is markovian with transition probability P. The theory of Markov chains describes some interesting properties of this kinematics that are relevant to the problem of true perception. We describe these properties informally now, and formally in the next chapters.

Depending on the details of the transition probability P, one finds that the state space E of the markovian dynamics contains different "pockets" which act like traps; if the state of the chain happens to enter one of these pockets, then the chain will forever stay within that pocket almost surely. For this reason these pockets are called "absorbing sets." The complement in the state space of all the absorbing sets is a pool of states called the "transient states." This is depicted in Figure 5.1, where the white disks represent absorbing sets

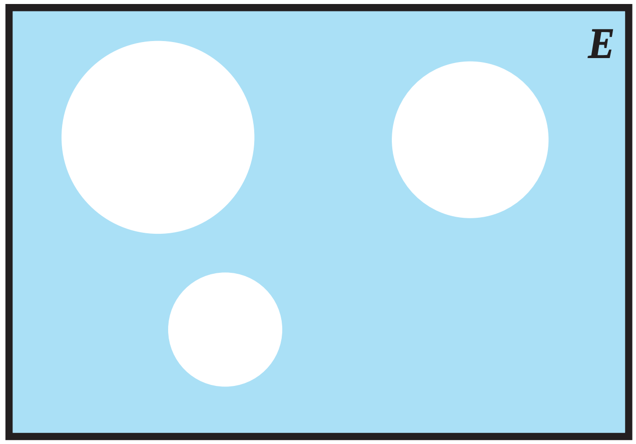

FIGURE 5.1. *Absorbing sets on the state space of a markovian dynamics. White disks represent absorbing sets. Blue regions represent transient states.*

and the states outside the disks are the transient states. An absorbing set may contain infinitely many states. If a chain enters an absorbing set, the chain then marches probabilistically from state to state within the absorbing set, and almost surely never enters a state outside of the absorbing set.

One finds that, for each absorbing set C, there is a unique probability measure supported on C which describes the long term behavior of the chain, once it is trapped in C. This measure, say m, gives for each subset D of the absorbing set a probability, $m(D)$; $m(D)$ can be interpreted as the relative frequency that the trapped chain is found within D over a very long time. The measure m is called a "stationary" measure; an example of such a measure for a dynamics of two participators is shown in Figure 5.2. Darker regions indicate higher frequency states. The little circles drawn over the stationary measure indicate the perspectives each participator happens to adopt at some instant of the dynamics.[4]

Now if a two participator dynamical chain enters an absorbing set with stationary measure m, then each participator can reach true perceptual conclu-

[4] Figure 5.2 does not represent the stationary measure on the original state space of the Markov chain. The original state space is a product space, E^2, where there are two participators in the chain. Figure 5.2 represents the stationary measure on a single copy of E, which describes the perspective of B *relative* to A.

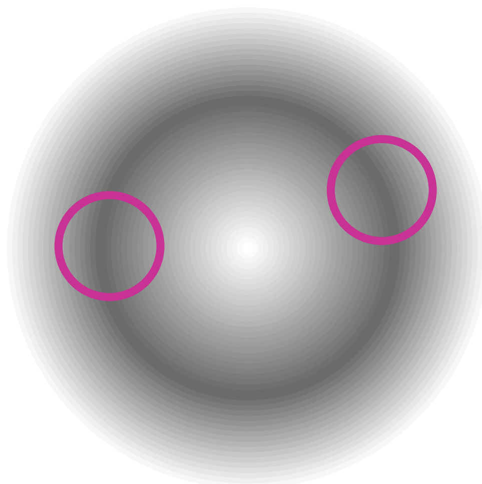

FIGURE 5.2. *A stationary measure. Darker regions indicate higher probability states.*

sions if its conclusions η are related appropriately to m. That is, a participator perceives truly if its perceptual conclusions η are matched to the dynamical reality observed, namely m. The way to match η to m is to make the measures $\eta(s, \cdot)$ the appropriate conditional probability measures of m, as depicted in Figure 5.4. This figure shows the stationary measure m of the dynamics of one participator relative to another, where the latter's perspective is always taken to be the origin at each step. At the instant shown, the two participators are channeling, leading the participator at the origin to have premise s. It can be seen that the appropriate conclusion η for this premise is the conditional probability of m when m is restricted to the line between the participators, viz., the line $\pi^{-1}(s)$. By choosing $\eta(s, \cdot)$ to be this probability measure, the participator at the origin has its perceptual conclusions matched to reality. Thus, in the case of a two participator dynamics involving only cross-channelings, the equation that specifies when perception matches reality simply asserts that the conclusion kernel η is the rcpd with respect to π of a stationary measure m. A measure m is stationary under the action of the transition probability P if $m = mP$, i.e., if

$$m(I - P) = 0, \tag{5.3}$$

where I is the identity operator. In the dynamics considered here, this equation, together with the stipulation that η is the rcpd of m, is the "perception = reality" equation. Note that there are in general many absorbing sets, each with its own stationary measure, so that the measure m is not uniquely determined even when P is fixed. Therefore, to determine if perception matches reality, we must be careful to use the appropriate stationary measure.

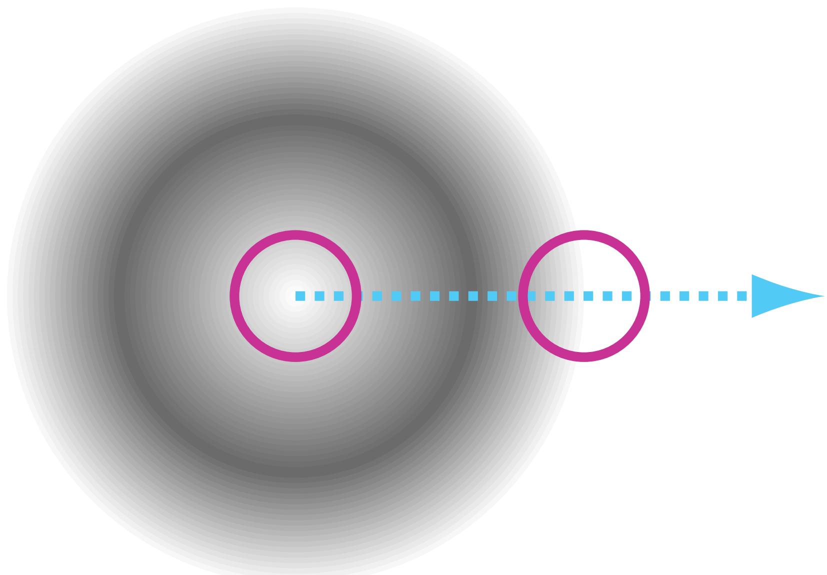

FIGURE 5.4. *A participator's conclusion measure should be derived as the rcpd of the appropriate stationary measure.*

Now if the chain never enters an absorbing set, i.e., if the dynamics is not stable, then there is no stationary probability measure to use to compute η. True perception is not possible. There are no probability measures $\eta(s, \cdot)$ that are matched to the dynamical reality. We see that a *stable* dynamics of perspectives is necessary for true perception.

In chapter ten we discuss how, to each absorbing set, there are associated in a natural manner complex-valued eigenfunctions of the transition probability P. We show that the squared amplitude of these eigenfunctions yields a probability measure which is stationary or *asymptotic* (a property, to be discussed later, which is slightly weaker than stationarity).

6. An example

We close this chapter with an illustration of participator dynamics by means of a specific and elementary example, including a computation of its stationary measures. Consider the symmetric framework (X, Y, E, S, G, J, π), where

$$\begin{aligned} X &= \mathbf{R}, \qquad E = \mathbf{Z} \text{ (the integers)}, \\ Y &= S = \{1, 0, -1\},\ G = \langle \mathbf{R}, + \rangle,\ J = \langle \mathbf{Z}, + \rangle, \\ \pi(x) &= \operatorname{sgn}(x) \end{aligned} \tag{6.1}$$

and where the signum function "sgn" is given by

$$\operatorname{sgn}(x) \quad = \quad \begin{cases} 1, & \text{if } x > 0; \\ 0, & \text{if } x = 0; \\ -1, & \text{if } x < 0. \end{cases} \tag{6.2}$$

Suppose we have two participators labelled "1" and "2" respectively, which channel with each other at each instant of reference time. As before, we do not allow self-channeling. Both participators are assumed to have the same action kernel Q, defined as follows:

$$Q(0, \cdot) \quad = \quad \epsilon_0(\cdot)$$

(where $\epsilon_0(\cdot)$ is Dirac measure at 0); if $r \neq 0$,

$$Q(r, x) \quad = \quad \begin{cases} \rho, & \text{if} \quad x = \operatorname{sgn}(r); \\ 1 - \rho, & \text{if} \quad x = \operatorname{sgn}(-r); \\ 0, & \text{otherwise.} \end{cases} \tag{6.3}$$

(Here r is the relative position before channeling and x is the participator's change in position after channeling; we assume that the quantity ρ lies between 0 and 1). In words: if a channeling came from the participator's current position, there is no movement. Otherwise, the participator moves one step in the direction from which the channeling came, with a probability of ρ, or one step away from that direction, with the complementary probability of $1 - \rho$.

Imagine that the two participators are initially separated by a nonzero distance. After they channel, their relative distance will either remain unchanged, or will have changed by two units. These are the only possibilities. (If they were initially at the same position in E, nothing will change.) This is expressed in the following derivation of the dynamical kernel P of the joint markovian dynamics (as introduced in section four above). Note that the dynamics is ***relativized***; it is a dynamics on the group **Z** of the relative displacements of participator 2 with respect to participator 1.

$$1-\rho \leftarrow \bullet \rightarrow \rho \qquad\qquad\qquad\qquad \rho \leftarrow \bullet \rightarrow 1-\rho$$

$$* * *| * * * *| * * * * \bigcirc * * * * | * * * *| * * * *| * * * *| * * * * \bigcirc * * * * | * * * *| * * *$$

Participator 2 Participator 1

FIGURE 6.4. *A markovian two-participator dynamics with $E = \mathbf{Z}$. The current relative separation is $r = -5$. After a channeling each participator will jump in the indicated directions with the given probabilities.*

Proposition 6.5. Let r denote the current relative separation and q the relative separation after channeling. Then the kernel P of the dynamics is given by
If $r = 0$,

$$P(0, q) = \epsilon_0(q). \tag{6.6}$$

If $q = r$, $r \neq 0$,

$$P(r, q) = 2\rho(1-\rho). \tag{6.7}$$

If $q = r - 2\mathrm{sgn}(r)$, $r \neq 0$,

$$P(r, q) = \rho^2. \tag{6.8}$$

If $q = r + 2\mathrm{sgn}(r)$, $r \neq 0$,

$$P(r, q) = (1-\rho)^2. \tag{6.9}$$

If $q \neq r$, and $q \neq r \pm 2$

$$P(r, q) = 0. \tag{6.10}$$

Proof. The result is a consequence of the assumption of independence between the jumps of the individual participators, as expressed in Proposition 4.5. By that Proposition we see that

$$\begin{aligned} P(r,q) &= [Q,Q](r,q) \\ &= \sum_{z,w} Q(r,z)Q(-r,w)1_q(r-z+w) \\ &= \sum_w Q(r,(r-q)+w)Q(-r,w). \end{aligned} \tag{6.11}$$

The result then follows from the definition 6.3 of Q, after analyzing the possibilities into the indicated cases. ∎

Notice that $\sum_q P(r,q) = 1$, for all $r \in \mathbf{Z}$.

Up to an arbitrary initial probability measure ξ on the group $\mathbf{Z}$, we have described the Markov chain which is the (relative) dynamics. We may now inquire into the long-term behavior of the dynamics, as introduced in section five.

Suppose that ν is a probability measure on $\mathbf{Z}$. Recall that ν is said to be ***stationary*** for the chain with T.P. P if

$$\nu P = \nu,$$

i.e., if for all $q \in \mathbf{Z}$,

$$\sum_r \nu(r)P(r,q) = \nu(q).$$

This is just equation 5.3 transcribed to our situation. For convenience we extract the $r = 0$ term in the sum on the left, to get

$$\sum_{r \neq 0} \nu(r)P(r,q) + \nu(0)\epsilon_0(q) = \nu(q). \tag{6.12}$$

If $\rho = 1$, the participators simply move towards each other after any channeling. Imagine that the participators are initially an even distance apart. Then they will move towards each other until they are at the same point, thenceforth to remain there. If they were to start an odd distance apart, they would eventually find themselves one unit apart. From then on they would oscillate, with relative positions of ± 1. Thus when $\rho = 1$ there are two stationary measures: Dirac measure $\epsilon_0(\cdot)$ at the origin and a measure μ given by $\mu(+1) = \mu(-1) = 1/2$, $\mu(q) = 0$ if $q \neq \pm 1$.

In general, the set of measures stationary with respect to a given T.P. is always a convex set. That is, if λ and σ are stationary, so is $a\lambda + b\sigma$ whenever $0 \leq a, b \leq 1$ and $a + b = 1$. In particular, in our situation when $\rho = 1$ the set of stationary measures consists of all convex combinations $a\epsilon_0(\cdot) + b\mu(\cdot)$.

Note that, regardless of the value of ρ, $\nu(\cdot) = \epsilon_0(\cdot)$ is always a stationary measure for P. It is interesting that the only set of values of ρ for which the dynamics has a stationary measure other than $\epsilon_0(\cdot)$ is the interval $(\frac{1}{2}, 1]$. In the rest of this section we will demonstrate this fact and explicitly determine the stationary measures.

If $\rho = 0$ it is intuitively clear from 6.3 that the chain wanders off to infinity, if it is not already at the origin. Thus, if $\rho = 0$, the Dirac measure at zero is in fact the only stationary measure. Henceforth we assume $\rho \neq 0$.

Now applying Proposition 6.5 to equation 6.12, we identify the following cases:

(i) If $q = 0$,

$$\nu(0) = \rho^2(\nu(2) + \nu(-2)) + \nu(0). \tag{6.13}$$

(ii) If $q = \pm 1$,

$$\nu(\pm 1) = 2\rho(1-\rho)\nu(\pm 1) + \rho^2\nu(\mp 1) + \rho^2\nu(\pm 3). \tag{6.14}$$

(iii) If $q = \pm 2$,

$$\nu(\pm 2) = 2\rho(1-\rho)\nu(\pm 2) + \rho^2\nu(\pm 4). \tag{6.15}$$

These cases are special; for the general case $|q| \geq 3$, we have

$$\nu(q) = 2\rho(1-\rho)\nu(q) + \rho^2\nu(q + 2\mathrm{sgn}(q)) + (1-\rho)^2\nu(q - 2\mathrm{sgn}(q)),$$

which, with a little algebra, may be re-expressed as follows:

(iv) If $|q| \geq 3$,

$$\nu(q) = c^2\nu(q + 2\mathrm{sgn}(q)) + s^2\nu(q - 2\mathrm{sgn}(q)). \tag{6.16}$$

where

$$c^2 = \frac{\rho^2}{\rho^2 + (1-\rho)^2}, \qquad s^2 = \frac{(1-\rho)^2}{\rho^2 + (1-\rho)^2}; \tag{6.17}$$

note that $c^2 + s^2 = 1$.

Equation 6.16 is a linear difference equation with constant coefficients. Its solutions may be obtained by substituting the trial solution $\nu(q) = x^q$, $x \neq 0$.

Doing so, we get

$$x^q = c^2 x^{q+2\mathrm{sgn}(q)} + s^2 x^{q-2\mathrm{sgn}(q)}, \qquad |q| \geq 3 \tag{6.18}$$

Now the substitution $x \to x^{-1}$ into 6.18 converts any solution for $q \geq 3$ into one for $q \leq -3$, as may easily be checked. This allows us to concentrate on 6.18 for positive q only. So doing, and dividing out by x^{q-2}, we arrive at the characteristic equation

$$c^2 x^4 - x^2 + s^2 = 0. \tag{6.19}$$

Solving this for x^2, we get $x^2 = 1$ or $(s/c)^2$. (A quick way to see this is to set $c = cos\,\theta$ and $s = sin\,\theta$ and to use elementary trigonometric formulas.)

If $s = c$, these two solutions to 6.19 are the same. This happens when $\rho = 1/2$. For the moment, assume $s \neq c$. Put

$$t = \left(\frac{s}{c}\right)^2 = \left(\frac{1-\rho}{\rho}\right)^2. \tag{6.20}$$

We may immediately solve 6.16 for ν at the even integers. If $\rho \neq 1/2$ (i.e., $t \neq 1$), every solution to 6.16 is, at even values of q, of the form

$$\nu(2k) = \begin{cases} a_+ + b_+ t^k, & \text{if } k \geq 2 \\ a_- + b_- t^{|k|}, & \text{if } k \leq -2 \end{cases} \qquad (t \neq 1) \tag{6.21}$$

for some constants $a_\pm, b_\pm$.

Consider now $t = 1$. Then $s^2 = c^2 = 1/2$ and, by 6.16, $\nu(q)$ is an average of $\nu(q+2)$ and $\nu(q-2)$:

$$\nu(q) = \tfrac{1}{2}\nu(q+2) + \tfrac{1}{2}\nu(q-2).$$

The characteristic equation of this difference equation is

$$x^4 - 2x^2 + 1 = 0,$$

so that x^2 can only be unity. In this case, we have

$$\nu(2k) = \begin{cases} a_+ + b_+ k, & \text{if } k \geq 2 \\ a_- + b_- |k|, & \text{if } k \leq -2 \end{cases} \qquad (t = 1), \tag{6.22}$$

for some constants $a_\pm$ and $b_\pm$, as the general solution of 6.16.

Lemma 6.23. If ν is a stationary measure with respect to the T.P. P of Proposition 6.5, then

$$\nu(2k) = 0 \quad \text{for all } k \in \mathbf{Z},\ k \neq 0.$$

Proof: Since ν is a probability measure, $\{\nu(2k)\}_{k=0}^{\infty}$ is a summable sequence of non-negative terms. Hence $a_+ = a_- = 0$. If $\rho = 1$ (i.e., $t = 0$ by 6.20), by 6.21 we are done.

Next assume that $0 < t \neq 1$. By 6.13 we have $\rho^2(\nu(2) + \nu(-2)) = 0$. But, since $\rho \neq 0$, the non-negative quantities $\nu(2)$ and $\nu(-2)$ are both null. By 6.15, the same holds for $\nu(4)$ and $\nu(-4)$. When $k = 2$, 6.21 says

$$0 = \nu(4) = 0 + b_+ t^2$$

$$0 = \nu(-4) = 0 + b_- t^2,$$

that is, $b_+ = b_- = 0$. Thus the result obtains if $t \neq 1$.

If $t = 1$, the same requirement of summability shows, using 6.22, that only $\nu(0)$ could possibly be nonzero. ∎

We turn now to the computation of ν at odd integral points. Assume that $\rho \neq \frac{1}{2}$ (i.e., $t \neq 1$). We solve the formal difference equation in 6.16 for ν at the odd integers. The general solution has the form

$$\begin{aligned} \nu(2k+1) &= c_+ + d_+ t^{|k|}, \qquad \text{if } k \geq 0; \\ \nu(2k-1) &= c_- + d_- t^{|k|}, \qquad \text{if } k \leq 0, \end{aligned} \tag{6.24}$$

for some constants c_+, d_+, c_-, and d_-. As in the even case, summability requires that $c_+ = c_- = 0$ and that $t < 1$. Thus, in terms of $q = 2k+1$ (for $k \geq 0$) or $q = 2k-1$ (for $k \leq 0$), our general solution is,

$$\text{for } \rho \neq \frac{1}{2} \qquad \nu(q) = \begin{cases} d_+ t^{|q-1|/2}, & \text{for odd } q \geq 0; \\ d_- t^{|q+1|/2}, & \text{for odd } q \leq 0. \end{cases} \tag{6.25}$$

In particular,

$$\nu(1) = d_+, \qquad \nu(-1) = d_-. \tag{6.26}$$

Since $\sum_q \nu(q) = 1$, we have that $\sum_{q \,\mathrm{odd}} \nu(q) \leq 1$. Thus

$$\sum_{k \leq 0} d_- t^{|k|} + \sum_{k \geq 0} d_+ t^{|k|} = \frac{d_+ + d_-}{1-t} \leq 1. \tag{6.27}$$

In terms of ρ (using the definition 6.20 of t), this says that

$$1 \geq \rho \geq \frac{1}{1+\sqrt{1-(d_+ + d_-)}} \tag{6.28}$$

(which restricts ρ to the interval $(\frac{1}{2}, 1])$. We know that $\nu(2k) = 0$ if $k \neq 0$ (Proposition 6.23). Thus, by 6.27,

$$\nu(0) + \frac{d_+ + d_-}{1-t} = 1$$

or

$$\nu(0) = \frac{2\rho - 1 - \rho^2(d_+ + d_-)}{2\rho - 1}, \qquad \frac{1}{2} < \rho \leq 1. \tag{6.29}$$

We are now in a position to delineate all possible stationarities of this chain. This is significant, for once we know the stationary measures it is possible to describe the "true perception" of the dynamical situation by a given participator, as discussed in the previous section of this chapter. We shall not delve into such detail here; our purpose is to give a feel for how the dynamics is analyzed. We end this chapter with the following theorem.

Theorem 6.30.

(i) If $\frac{1}{2} < \rho \leq 1$, there is a one-parameter family of probability measures stationary with respect to the T.P. P (given in 6.5) of the dynamical chain of our example. With parameter denoted by d, this family may be described as:

$$\nu(q) = \begin{cases} d\left(\dfrac{1-\rho}{\rho}\right)^{|q-1|}, & \text{if } q \text{ is odd and } q > 0; \\ d\left(\dfrac{1-\rho}{\rho}\right)^{|q+1|}, & \text{if } q \text{ is odd and } q < 0; \\ 0, & \text{if } q \text{ is even and } q \neq 0; \\ \dfrac{2\rho - 1 - 2\rho^2 d}{2\rho - 1}, & \text{if } q = 0. \end{cases}$$

The range of allowed values of the parameter d is contained in the closed interval $[0, 1]$. For fixed $\rho \in (\frac{1}{2}, 1]$ the range is $[0, (2\rho - 1)/2\rho^2]$.

(ii) If $0 \leq \rho \leq \frac{1}{2}$, the only stationary measure is $\epsilon_0(\cdot)$.

Proof. Consider (i). For q odd we have equation 6.25. Recalling from 6.20 that $t^{1/2} = (1-\rho)/\rho$, we obtain the first two formulas below.

$$\nu(q) = \begin{cases} d_+ \left(\dfrac{1-\rho}{\rho}\right)^{|q-1|}, & \text{if } q \text{ is odd and } q > 0; \\ d_- \left(\dfrac{1-\rho}{\rho}\right)^{|q+1|}, & \text{if } q \text{ is odd and } q < 0; \\ 0, & \text{if } q \text{ is even and } q \neq 0; \\ \dfrac{2\rho - 1 - \rho^2(d_+ + d_-)}{2\rho - 1}, & \text{if } q = 0. \end{cases}$$

The third formula above is Lemma 6.23 and the fourth is equation 6.29.

Substituting the formula for odd q into 6.14 we get

$$d_\pm = 2\rho(1-\rho)d_\pm + \rho^2 d_\mp + \rho^2 d_\pm \left(\frac{1-\rho}{\rho}\right)^{|\pm 2|},$$

which reduces to $d_+ = d_-$. Set $d = d_+ = d_-$; the range of allowed values of the parameter d as given in the statement is computed by requiring that

$$0 \leq \nu(0) = \frac{2\rho - 1 - 2\rho^2 d}{2\rho - 1} \leq 1.$$

This concludes (i).

It remains to verify (ii). We have already done so for $0 < \rho < 1/2$, since 6.28 shows us that the fact that ν is a probability measure requires that $\rho \geq 1/2$. Moreover, the instance $\rho = 1/2$ requires, in the same way as in 6.22 above, that

$$\nu(2k+1) = \begin{cases} a_+ + b_+ k, & \text{if } k \geq 1 \\ a_- + b_-|k|, & \text{if } k \leq -1 \end{cases}$$

which is only summable if it is in fact zero. ∎